CONSTRUCTIVE NONLINEAR CONTROL: A HISTORICAL PERSPECTIVE *

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Abstract

In the early days of nonlinear control theory most of the stability, optimality and uncertainty concepts were descriptive rather than constructive. This survey describes their 'activation' into design tools and constructive procedures. Structural properties of nonlinear systems, such as relative degree and zero dynamics, are connected to passivity, while dissipativity, as a finite $\mathcal{L}_2$-gain property, also appears in the disturbance attenuation problem, a nonlinear counterpart of robust linear control. Passivation-based designs exploit the connections between passivity and inverse optimality, and between Lyapunov functions and optimal value functions. Recursive design procedures, such as backstepping and forwarding, achieve certain optimal properties for important classes of nonlinear systems. The survey concludes with four representative applications. The selection of the topics and their interpretations are greatly influenced by the experience and personal views of the senior author.

Key words: design tools, robust stabilization, inverse optimality, feedback passivation, recursive procedures, applications

1 INTRODUCTION

Nonlinear feedback control has been the topic of hundreds of publications, numerous monographs and several comprehensive textbooks, such as Khalil

(1996b), Vidyasagar (1993), Sastry (1999). In reviewing this wealth of information severe and unfair omissions are inevitable. This survey will follow a personal path and discuss some developments in which the first author was a participant or, at least, a curious bystander. It begins with an era that was formative for most of stability, optimality and uncertainty concepts. These concepts were more *descriptive* than *constructive*: they were used to describe system properties rather than to design a system which will possess these properties.

The main part of the survey describes, in broad brush strokes, the ongoing 'activation process', through which some of the earlier descriptive concepts are being converted into design tools within constructive procedures applicable to common classes of nonlinear systems. This process is a confluence of several research streams. Differential-geometric concepts describe structural properties of nonlinear systems, such as relative degree and zero dynamics. These properties suggest a connection with passivity, while dissipativity, as a finite $\mathcal{L}_2$-gain property also appears in the disturbance attenuation problem, treated in the dynamic game framework. This is a nonlinear counterpart of robust linear control, which itself is closely related to dissipativity through the fundamental lemmas on passivity and boundedness. Passivity is a key concept in the inverse problem of optimal control, which reveals a connection between Lyapunov functions and optimal value functions as solutions of the Hamilton-Jacobi equation.

2 DESCRIPTIVE CONCEPTS

In the 1940's-50's, control theory in the East was influenced by mechanics, while in the West it emerged from the Nyquist-Bode feedback theory for active filters. Initially, these two cultures spoke different languages: the state space language in the East, and the input-output language in the West. The First IFAC Congress in Moscow, 1960, brought the two cultures together to create today's 'bilingual' control theory.

2.1 Lyapunov Stability

Stability concepts formulated by Lyapunov at the end of the last century were advanced by Malkin (1952), Chetaev (1955), Zubov (1957), Krasovskii (1959), and surveyed by Kalman and Bertram (1960), LaSalle and Lefschetz (1961), Lefschetz (1965), and Hahn (1967). These advances included various *converse* and *invariance theorems* by Massera (1956), Kurzweil (1956), Krasovskii (1959), Yoshizawa (1966), and LaSalle (1968), which are frequently

used today.

The effects of *persistent disturbances* were analyzed by Malkin (1952), Krasovskii (1959), and Hahn (1967), who used the terms *practical stability* or *total stability* to describe boundedness under small perturbations. Systems in which switching controls (*variable structure systems*) eliminate the effects of disturbances by introducing *sliding modes*, were investigated by Filippov (1964), Barbashin (1967), Emelyanov (1967), Filippov (1988), and Utkin (1992). *Vector Lyapunov functions* introduced by Bellman and Matrosov were applied to large scale systems by Michel and Miller (1977), and Šiljak (1978).

2.2 Absolute Stability and the PR Lemma

For a long time a serious drawback of Lyapunov theory was the lack of procedures for construction of Lyapunov functions. Among the early attempts to remove this drawback, the *absolute stability* approach of Lurie (1951), as presented in Aizerman and Gantmacher (1964), remained highly influential. For systems consisting of a linear block in feedback with a static nonlinearity, Lurie and coworkers derived algebraic equations for Lyapunov functions made of a quadratic form and the integral of the nonlinearity. The first challenge posed by the absolute stability problem was to characterize those linear blocks for which such *quadratic-plus-integral* functions exist, given that the nonlinearity belongs to a known sector. The second challenge was to provide a procedure for solving the algebraic equations.

In response to these challenges, many blind alleys were explored for a decade. Then suddenly, the absolute stability problem was solved with a frequency domain criterion by Popov (1960, 1962), which was an instant success. Its state space form was soon established in a lemma by Yakubovich (1962) and Kalman (1963). From today's standpoint, the fundamental contribution of Popov's criterion is the introduction of the concept of passivity (positive realness) in feedback control. The crucial positive real (PR) property was made explicit by Popov (1963) and, independently, by Brockett (1964). The lemma of Yakubovich and Kalman was subsequently named the *Positive Real Lemma*. For a minimal state space realization (A, B, C) of the transfer function $H(s)$, the PR Lemma shows that $ReH(j\omega) \geq 0$ is equivalent to the existence of a $P = P^T > 0$ such that

$$A^T P + PA \leq 0 \quad \text{and} \quad B^T P = C. \tag{1}$$

Thus, $H(s)$ being passive means that P satisfies not only the Lyapunov inequality, but also an input-output constraint which restricts the relative degree of $H(s)$ to be zero or one, and its zeros to be stable (minimum phase). Matrix

3

P defines the quadratic form in the quadratic-plus-integral Lyapunov function and, hence, any procedure that solves (1) can be used to construct this function. Extensions and interpretations of the PR Lemma were given by Anderson (1967), Anderson and Vongpanitlerd (1973), Narendra and Taylor (1973), and more recently, by Tao and Ioannou (1988), Wen (1988), Lozano-Leal and Joshi (1990), Ioannou and Sun (1996), Rantzer (1996), and Xiao and Hill (1998).

Popov's work also led to several practically appealing *circle criteria* by Narendra and Goldwyn (1964), Sandberg (1964a), Zames (1964, 1966), Naumov and Tsypkin (1965), Yakubovich (1965), Brockett and J.L.Willems (1965), Narendra and Neuman (1966), Cho and Narendra (1968), Zames and Falb (1968), and others, insightfully surveyed by Brockett (1966), and treated in detail in the book by Narendra and Taylor (1973). Among these results, particularly important are the *multiplier* methods, which paved the road for development of modern robust control with structured uncertainty. A recent unified treatment with further advances are presented by Megretski and Rantzer (1997).

Tsypkin (1962, 1964, 1963, 1965) was among the first to recognize the fundamental importance of Popov's work. He formulated two absolute stability criteria for discrete-time systems. Further results in this direction were obtained by Jury and Lee (1964) and, more recently, by Kapila and Haddad (1996), and Park and Kim (1998). The discrete-time analog of the PR Lemma was derived by Kalman and Szegö (1963), Szegö (1963), and Hitz and Anderson (1969).

2.3 Passivity and Small-Gain Theorems

Following different paths, Popov (1963) and Zames (1966) formulated the fundamental and far-reaching *passivity theorem* stating that the feedback interconnection of two nonlinear passive blocks H_1 and H_2 is passive (see Figure 1). Sandberg (1964b) and Zames (1966) also formulated a *small-gain theorem*
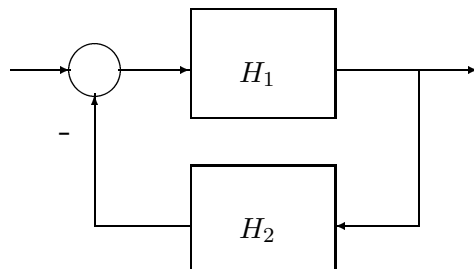

Fig. 1. Passivity and small-gain.

for closed-loop stability when the operator gain of H_1 connected with H_2 is less than one. Zames saw these small-gain and passivity theorems as nonlinear generalizations of the linear gain and phase results in the Nyquist-Bode

theory. His words are as enlightening today as they were then:

"The classical definitions of gain and phase shift, in terms of frequency response, have no strict meaning in nonlinear or time-varying systems. However, stability does seem to depend on certain measures of signal amplification and signal shift. Thus the norm ratio $|Hx|/|x|$ plays a role similar to the role of gain. Furthermore, the inner product (x, Hx), a measure of input-output cross-correlation, is closely related to the notion of phase shift. For example, for linear time-invariant operators the condition of positivity, $(x, Hx) \geq 0$, is equivalent to the phase condition, $|Arg\{H(j\omega)\}| \leq 90°$. Theorem 1 can be viewed as a generalization to nonlinear time-varying systems of the rule that, 'if the open-loop gain is less than one, then the closed-loop is stable.' Theorem 3 can be viewed as the generalization of 'if the open-loop absolute phase shift is less than 180° then the closed loop is stable.'"

Until the end of the 1980's, the passivity theorem was used primarily in adaptive control. The Sandberg-Zames small-gain theorem, refined by Desoer and Vidyasagar (1975), found a wide variety of applications, including robust linear control with bounded norm uncertainty. A nonlinear sector formulation and a unified treatment of small-gain and passivity theorems were pursued by Safonov (1980), Hill and Moylan (1980*b*, 1983), and Teel *et al.* (1996).

2.4   Lyapunov Functions and Dissipativity

The PR Lemma connected passivity with the quadratic-plus-integral Lyapunov functions for the Lurie class of systems. For more general nonlinear systems, such a connection was made by Willems (1972) with the theory of dissipative systems, extended by Hill and Moylan (1977, 1980*a,b*). For a system H with state x, input u and output y, Willems introduced a *storage function* $S(x) \geq 0$, $S(0) = 0$, and a *supply rate* $w(u, y)$ and defined H as *dissipative* if $\dot{S}(x(t)) \leq w(u(t), y(t))$.[1] Passivity is the special case when $w = u^T y$. An analogy of storage S is the system energy, and the supply rate w is analogous to the power delivered to the system by the external sources. Dissipativity with the supply rate

$$w(u, y) = u^T y - \rho y^T y - \nu u^T u \,, \tag{2}$$

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[1] The integral form of this definition does not require S to be differentiable, only w to be integrable. Henceforth, the important issue of differentiability will not be discussed. The lack of differentiability requires more general solution concepts for various PDE's in robust nonlinear control. Best known among them is the *viscosity solution* by Crandall *et al.* (1984).

can be used to quantify the *excess* or *shortage* of passivity via ρ and ν. In a feedback loop, a positive or negative 'amount' of passivity can be reallocated from the feedforward to the feedback path, or vice-versa, using *loop transformations*, already suggested by Popov and Zames. Moylan (1974), and Hill and Moylan (1976) extended the PR Lemma by showing that the nonlinear system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x) + j(x)u, \qquad x \in {I\!\!R}^n, \ u, y \in {I\!\!R}^m,
\end{aligned} \tag{3}$$

is dissipative with the supply rate (2) if and only if there exist functions $S(x) \geq 0$, $q(x)$ and $W(x)$ such that

$$\begin{aligned}
L_f S(x) &= -\frac{1}{2} q^T(x) q(x) - \rho h^T(x) h(x) \\
L_g S(x) &= h^T(x) - 2\rho h^T(x) j(x) - q^T(x) W(x) \\
W^T(x) W(x) &= -2\nu I + j(x) + j^T(x) - 2\rho j^T(x) j(x),
\end{aligned} \tag{4}$$

where $L_f S(x) := \frac{\partial S}{\partial x} f(x)$ and $L_g S(x) := \frac{\partial S}{\partial x} g(x)$. This is the nonlinear analog of a PR Lemma more general than (1). In the special case of passivity we have $\rho = 0$ and $\nu = 0$. If the throughput is absent, that is $j(x) = 0$, condition (4) reduces to

$$L_f S \leq 0, \quad (L_g S)^T = h(x), \tag{5}$$

which is the exact analog of (1). If $S(x)$ is positive definite, it can be taken as a Lyapunov function which connects dissipativity and passivity with stability properties of the system (3). A closely related result, *Bounded Real Lemma*, has also been extended to nonlinear systems by Hill and Moylan (1976). It played an important part in the development of nonlinear $\mathcal{H}_\infty$ control.

2.5 Optimal and Inverse Optimal Control

To improve performance, we often try to find a feedback control $u(x)$ that stabilizes the system (3) while minimizing the cost

$$J = \int_0^\infty (l(x) + u^T R(x) u) dt, \tag{6}$$

with $l(x) \geq 0$ and $R(x) > 0$ for all x. A glimpse into the 1950-1960 efforts to solve such optimal control problems can be gained from the textbooks by

Athans and Falb (1965), Lee and Markus (1967) and Anderson and Moore (1971). If $V(x) \geq 0$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$L_f V(x) - \frac{1}{4} L_g V(x) R^{-1}(x)(L_g V(x))^T + l(x) = 0, \quad V(0) = 0, \quad (7)$$

then the optimal feedback law is

$$u = -\frac{1}{2} R^{-1}(x)(L_g V(x))^T, \quad (8)$$

and $V(x)$ is its *value function*, that is, the minimum value of J for the initial state x. Under a detectability condition with $l(x)$ as the system output, the optimal control (8) is stabilizing. Furthermore, if the value function $V(x)$ is positive definite, it can be used as a Lyapunov function, thus establishing a connection between stability and optimality, as discussed in the books by Sepulchre *et al.* (1997) and Sontag (1998b).

In the *inverse optimal control problem*, a Lyapunov function $V(x)$ is given and the task is to determine whether a control law such as (8) is optimal for a cost in the form (6). The control law $u = -(L_g V(x))^T$, referred to as $L_g V$-control, was studied by Zubov (1966), A. Krasovsky (1971), Jacobson (1977), Jurdjević and Quinn (1978), and other authors.

A connection between optimality and passivity for a linear system with a quadratic cost (the LQR problem) was established by Kalman (1964) who analyzed the inverse LQR problem. For the nonlinear system (3) and the cost (6) with $R(x) = I$, the passivity-optimality connection, made by Moylan and Anderson (1973), is that a control law $u = -\mu(x)$ is optimal if and only if the system $\dot{x} = f(x) + g(x)u$ with the input u and the output $y = \mu(x)$ is dissipative with the rate $w(u, y) \leq u^T y + \frac{1}{2} y^T y$. This means that the system is rendered passive not only with the unity feedback $u = -y$, but also with $u = -ky$ where $k \in [\frac{1}{2}, \infty)$, which is its gain margin. In this sense optimality enhances not only performance, but also robustness. Generalizations of stability margins were made by Anderson and Moore (1971), Safonov and Athans (1977), Safonov (1980) and Molander and Willems (1980). An analysis of stability margins for nonlinear optimal regulators was given by Glad (1984), and Tsitsiklis and Athans (1984).

2.6 Dynamic Games and Robust Control

Already in the 1960's it was clear that for robustness against disturbances and unmodeled dynamics various stability margins are insufficient, even in

linear systems. A general framework for worst case designs using dynamic (differential) games was introduced by Isaacs (1965). Their rapid development in the 1970's can be traced through the textbooks by Bryson and Ho (1969), Başar and Olsder (1982), Krasovskii and Subbotin (1988) and Krasovskii and Krasovskii (1995). Dorato and Drenick (1966) were the first to suggest that this dynamic game framework be employed for robust control. Some early attempts in this direction were made by Medanić (1967), Başar and Mintz (1972), Bertsekas and Rhodes (1971, 1973), and Mageirou (1976), to mention only a few. However, they haven't led to what we today call *robust control*, which instead was launched by Zames (1981) with an input-output formulation and $\mathcal{H}_\infty$ norms in the frequency domain. The development of $\mathcal{H}_\infty$ designs, which dominated most of the 1980's, is well known and, as a linear topic, is not within the scope of this survey.

To derive nonlinear counterparts of linear $\mathcal{H}_\infty$ results, most researchers had to return to state-space models, that is to dynamic games. This return, implicit in several linear results, including that of Doyle *et al.* (1989), was made explicit in the monograph by Başar and Bernhard (1995), which provided a rigorous foundation for robust nonlinear control and disturbance attenuation designs in the 1990's.

3   ACTIVATED CONCEPTS

Nonlinear concepts remained descriptive for a long time. Their 'feedback activation' began only recently, when some local properties were replaced with new concepts applicable to large regions of the state space. The main effort of activation is to make new concepts dependent on, and transformable by feedback control. A prominent example is the concept of *control Lyapunov function* whose derivative depends on the control and can be made negative by feedback. Another example is *feedback passivity*, that is, the possibility to render a system passive using feedback.

3.1   Input-to-State Stability

For systems with disturbances, Sontag (1989a) replaced the local notion of total stability with a more useful global concept of *input-to-state stability* (ISS). The system

$$\dot{x} = f(x, w), \quad f(0, 0) = 0 \tag{9}$$

8

is ISS if there exist a class-$\mathcal{KL}$ function[2] $\beta(\cdot, \cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that

$$|x(t)| \leq \max \left\{ \beta(|x(0)|, t), \gamma \left(\sup_{0 \leq \tau \leq t} |w(\tau)| \right) \right\}. \tag{10}$$

When the effect of the initial condition β vanishes as $t \to \infty$, the remaining term $\gamma(\cdot)$ is an *ISS-gain* of the system (9) from disturbance w to state x.

Sontag and Wang (1995) showed that the ISS property is equivalent to the existence of an *ISS-Lyapunov function*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{11}$$

such that

$$L_f V(x, w) \leq -\alpha_3(|x|) + \sigma(|w|), \tag{12}$$

where $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma(\cdot) \in \mathcal{K}$. An alternative characterization using $\alpha_4(\cdot), \rho(\cdot) \in \mathcal{K}$ is

$$|x| \geq \rho(|w|) \;\Rightarrow\; L_f V(x, w) \leq -\alpha_4(|x|). \tag{13}$$

Then, the ISS-gain $\gamma(\cdot)$ in (10) is the composition $\gamma(\cdot) = \alpha_1^{-1} \circ \alpha_2 \circ \rho(\cdot)$. A further refinement by Teel (1996*a*), and Sontag and Wang (1996) is the notion of *asymptotic gain* and its relation to ISS.

A small-gain theorem formulated by Hill (1991), and Mareels and Hill (1992), was extended in the ISS framework by Jiang *et al.* (1994), and further generalized by Teel (1996*a,b*). As an illustration, we quote an ISS small-gain result for the interconnected subsystems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, x_1). \end{aligned} \tag{14}$$

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[2] $\mathcal{K}$ is the class of functions $I\!\!R_{\geq 0} \to I\!\!R_{\geq 0}$ which are zero at zero, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of class-$\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $I\!\!R_{\geq 0} \to I\!\!R_{\geq 0}$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $I\!\!R_{\geq 0} \times I\!\!R_{\geq 0} \to I\!\!R_{\geq 0}$ which are class-$\mathcal{K}$ on the first argument and class-$\mathcal{L}$ on the second argument. The inverse of a class-$\mathcal{K}_\infty$ function exists and is also $\mathcal{K}_\infty$. The composition of class-$\mathcal{K}$ functions is also class-$\mathcal{K}$.

If the x_1-subsystem with x_2 as its input has ISS-gain $\gamma_1(\cdot)$, and the x_2-subsystem with x_1 as its input has ISS-gain $\gamma_2(\cdot)$, then the interconnection is globally asymptotically stable (GAS) if

$$\gamma_1 \circ \gamma_2(s) < s, \quad \forall s > 0. \tag{15}$$

A situation not covered by (10) is when the input $w(t)$ is unbounded, but has a finite energy norm. Sontag (1998a) defined the system (9) to be *integral input-to-state stable* (IISS) if there exist $\alpha(\cdot) \in \mathcal{K}_\infty$, $\beta(\cdot, \cdot) \in \mathcal{KL}$, and $\gamma(\cdot) \in \mathcal{K}$ such that, for all $t \geq 0$,

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|w(\tau)|)d\tau. \tag{16}$$

Angeli *et al.* (1998) showed that the IISS property is equivalent to the existence of an *IISS-Lyapunov function* which differs from the ISS Lyapunov function in that $\alpha_3(\cdot)$ in (12) is only positive definite, and not necessarily class-$\mathcal{K}_\infty$. While ISS implies IISS, the converse is not true: in the scalar system $\dot{x} = -\phi(x) + w$ with saturation $\phi(x) = \text{sgn}(x) \min\{|x|, 1\}$, the state $x(t)$ grows unbounded with the constant input $w(t) \equiv 2$, but it remains bounded if $\int_0^\infty |w(t)|dt$ exists, as shown by the IISS Lyapunov function

$$V(x) = \int_0^x \phi(s)ds \;\Rightarrow\; \dot{V} \leq -\phi(x)^2 + |w|.$$

3.2   Control Lyapunov Functions

The seemingly obvious concept of a *Control Lyapunov Function* (CLF) introduced by Artstein (1983) and Sontag (1983), made a tremendous impact on stabilization theory, which, at the end of the 1970's was stagnant. It converted stability descriptions into tools for solving stabilization tasks.

One way to stabilize a nonlinear system is to *select* a Lyapunov function $V(x)$ and then *try to find* a feedback control $u(x)$ that renders $\dot{V}(x, u(x))$ negative definite. With an arbitrary choice of $V(x)$ this attempt may fail, but if $V(x)$ is a CLF, we can find a stabilizing control law $u(x)$. For the nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{17}$$

$V(x)$ is a CLF if, for all $x \neq 0$,

$$L_g V(x) = 0 \quad \Rightarrow \quad L_f V(x) < 0. \tag{18}$$

By standard converse theorems, if (17) is stabilizable, a CLF exists. From (18), we see that the set where $L_gV(x) = 0$ is significant, because in this set the uncontrolled system has the property $L_fV(x) < 0$. However, if $L_fV(x) > 0$ when $L_gV(x) = 0$, then $V(x)$ is not a CLF and cannot be used for a feedback stabilization design (an observation that helps eliminate bad CLF candidates).

When $V(x)$ is a CLF, there are many control laws that render $\dot{V}(x, u(x))$ negative definite, one of which is given by a formula due to Sontag (1989b). The construction of a CLF is a hard problem, which has been solved for special classes of systems. For example, when the system is feedback linearizable we can construct for it a quadratic CLF in the coordinates in which the system is forced to become linear by a feedback transformation that cancels all the nonlinearities. Once such a CLF is constructed, it can be used to design a control law $u(x)$ that avoids cancelation of useful nonlinearities. For a larger class of systems CLF's can be constructed by *backstepping*, as discussed in Section 4.1.

3.3   CLF's for Systems with Disturbances

The CLF concept was extended by Freeman and Kokotović (1996a,b) to systems

$$\dot{x} = f(x, w) + g(x, w)u \,, \tag{19}$$

where w is a disturbance known to be bounded by $|w| \leq \Delta$, where Δ may depend on x. $V(x)$ is an RCLF (a *robust* CLF), if for all $|x| > c$, a control law $u(x)$ can be found to render $\dot{V}$ negative for any w such that $|w| \leq \Delta$. The value of c depends on Δ and on the chosen $u(x)$. For systems jointly affine in u and w,

$$\dot{x} = f(x) + g(x)u + p(x)w \,, \tag{20}$$

an 'activated' ISS-Lyapunov function, called ISS-CLF by Krstić *et al.* (1995), is a $V(x)$ for which a class-$\mathcal{K}_\infty$ function $\rho(\cdot)$ exists such that

$$|x| > \rho(|w|) \quad \Rightarrow \quad \exists u : \; L_fV(x) + L_pV(x)w + L_gV(x)u < 0 \,. \tag{21}$$

Again, the set $L_gV(x) = 0$ is critical because in it we require that

$$L_fV(x) + |L_pV(x)|\rho^{-1}(|x|) < 0 \,, \tag{22}$$

11

which means that $L_f V(x)$ must be negative enough to overcome the effect of disturbances bounded by $|w| < \rho^{-1}(|x|)$.

For systems with stochastic disturbances, Krstić and Deng (1998) introduced a notion of 'noise-to-state stability' (NSS) and the corresponding NSS-CLF convenient for this type of stabilization.

3.4 Disturbance Attenuation

The concepts of RCLF and ISS-CLF are closely related to the Hamilton-Jacobi-Isaacs (HJI) optimality conditions for dynamic games. For the system (19), a dynamic game is formulated by considering w as the maximizer and u as the minimizer of the cost

$$J = \int_0^\infty [q(x) + r(x, u)]dt, \tag{23}$$

where $q(x)$ and $r(x, u)$ penalize x and u in a meaningful way. In this formulation the disturbance w is not penalized. Instead, it is constrained by $|w| \leq \Delta$, where, as before, Δ may depend on x. If the *value* $V(x)$ of the associated game exists and is differentiable, then it satisfies the HJI equation

$$0 = \min_u \max_{|w| \leq \Delta} \{q(x) + r(x, u) + L_f V(x, w) + L_g V(x, w)u\}, \tag{24}$$

where the functions $L_f V$ and $L_g V$ depend on w through $f(x, w)$ and $g(x, w)$. The intractability of (24) motivated Freeman and Kokotović (1996a,b) to analyze an *inverse optimal robust control problem* in which $q(x)$ and $r(x, u)$ are not specified *a priori*. They derived conditions under which $V(x)$, constructed as an RCLF, is the value of a meaningful dynamic game, that is the solution of (24) for some $q(x)$ and $r(x, u)$ derived *a posteriori*, but *a priori* guaranteed to penalize both x and u. They further showed that, for (20), the *pointwise min-norm control law*

$$u_F(x) = \begin{cases} -\dfrac{\Psi(x)}{L_g V(x)} & \text{when } \Psi(x) > 0 \\ 0 & \text{when } \Psi(x) \leq 0, \end{cases} \tag{25}$$

where $\Psi(x) := L_f V(x) + |L_p V(x)|\Delta + \sigma(x)$ and $-\sigma(x) \leq 0$ is a 'margin of negativity', is inverse optimal for a meaningful class of penalties $q(x)$ and $r(x, u)$. The min-norm control law (25) was introduced earlier by Petersen and Barmish (1987).

As an illustration consider the cost (23) with $q(x) = x^2$ and $r(x, u) = u^2$ under the constraint $|w| \leq |x|$ for the system

$$\dot{x} = -x^3 + u + w \, , \tag{26}$$

where u is unconstrained. The optimal control

$$u^\star(x) = -x - x\sqrt{x^4 - 2x^2 + 2} + x^3 \, , \tag{27}$$

which satisfies the HJI equation, is 'intelligent': it vanishes for large $|x|$, when the term $-x^3$ is sufficient for robust stabilization. The inverse optimal control computed from (25) with $V(x) = \frac{1}{2}x^2$ and $\sigma(x) = x^2$ is

$$u_F(x) = \begin{cases} x^3 - 2x & \text{when } x^2 < 2 \\ 0 & \text{when } x^2 \geq 2 \, . \end{cases} \tag{28}$$

It is as 'intelligent' as the optimal control, because it becomes inactive for $x^2 \geq 2$, where $-x^3$ takes care of stabilization, (see Figure 2).
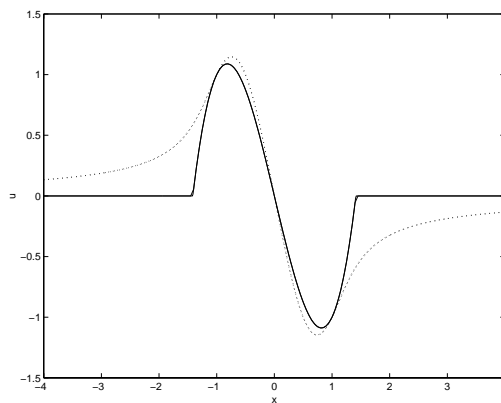

Fig. 2. $u^\star(x)$ -dotted, and $u_F(x)$- solid.

An analog of the linear $\mathcal{H}_\infty$ control is the *disturbance attenuation* problem extensively studied in the books by Başar and Bernhard (1995), Isidori (1995), van der Schaft (1996), Krstić and Deng (1998), Helton and James (1999) and in many papers including Ball and Helton (1992), Ball *et al.* (1993), Ball and van der Schaft (1996), Isidori and Astolfi (1992), Isidori and Kang (1995), James and Baras (1995), Krener (1994), and van der Schaft (1991, 1992). In most of these works the cost is

$$J = \int_0^\infty \left(|h(x)|^2 + |u|^2 - \gamma^2 |w|^2 \right) dt \, . \tag{29}$$

13

It can be verified (see van der Schaft (1996)) that the corresponding HJI equation yields a value function $V(x)$ and a control law $u^\star(x)$ which satisfy the Bounded Real Lemma of Hill and Moylan (1976) and, hence, the *dissipation inequality*

$$\dot{V} \le -|z|^2 + \gamma^2 |w|^2 \,, \tag{30}$$

with input w and output $z := (h(x), u^\star(x))$. Thus, as in the linear case, the $\mathcal{L}_2$-gain of the optimal closed-loop system is $\frac{\|z\|_2}{\|w\|_2} \le \gamma$.

However, for nonlinear systems, the use of the quadratic penalty $\gamma^2 |w|^2$ just to obtain an $\mathcal{L}_2$-gain has a disadvantage illustrated by the problem

$$\dot{x} = u + x^2 w \,, \tag{31}$$
$$J = \int_0^\infty (x^2 + u^2 - \gamma^2 w^2) dt \,,$$

for which the optimal control law

$$u^\star(x) = -\gamma \frac{x}{\sqrt{\gamma^2 - x^4}} \tag{32}$$

exists only for $x \in (-\sqrt{\gamma}, \sqrt{\gamma})$. Clearly, the disturbance w, which acts through x^2, is powerful when x^2 is large and the quadratic penalty $\gamma^2 w^2$ is insufficient to prevent the unboundedness of $x(t)$. This suggests that $\gamma^2 |w|^2$ in (29) be replaced by a class-$\mathcal{K}_\infty$ penalty function $\gamma(|w|)$ to be determined *a posteriori*. Krstić and Li (1998) constructed an ISS control law to be inverse optimal for a cost including $\gamma(|w|)$, illustrated again on the system (31). With $V = \frac{1}{2}x^2$ as an ISS-CLF, and $\rho(\cdot)$ in (21) taken to be $\rho(|w|) = |w|$, an ISS control law is $u = -(x^2 + \sqrt{x^4 + 1})x$. This control law satisfies the HJI condition with the cost

$$J = \int_0^\infty \left(\frac{2x^2}{x^2 + \sqrt{x^4 + 1}} + \frac{2u^2}{x^2 + \sqrt{x^4 + 1}} - \frac{27}{64}w^4 \right) dt \,. \tag{33}$$

Thus, for all x and all w the ISS property is achieved, but the optimality is with the penalty $\gamma(|w|) = \frac{27}{64}w^4$ rather than $\gamma^2 w^2$.

14

The disturbance attenuation problem for system (19) is more realistic when instead of the full state x, only an output y is assumed to be available

$$y = c(x) + v \,, \tag{34}$$

where v is the unknown measurement noise. The counterpart of cost (29) in this case is

$$J = \int_0^\infty (|h(x)|^2 + |u|^2 - \gamma^2 |w|^2 - \gamma^2 |v|^2) dt - N(x_0) \,, \tag{35}$$

where $N(x_0)$ is a positive definite cost on the unknown initial state. For this problem Didinsky and Başar (1992), Didinsky *et al.* (1993), Başar and Bernhard (1995) introduced the concept of a *cost-to-come* function $W(t, x)$, which is dual to the *cost-to-go* function $V(x)$ in traditional dynamic programming. Whereas $V(x)$ provides the evolution of the worst case cost from any time-state pair (t, x) into the future, $W(t, x)$ describes the worst cost from any time-state pair (t, x) back to the past, with the maximization taken over all disturbances w that are consistent with all the observations $y_{[0,t]}$ and controls $u_{[0,t]}$ up to time t.

As shown by Başar and Bernhard (1995), given $y_{[0,t]}$, $u_{[0,t]}$ and $x(t) = x$, the cost-to-come function satisfies the forward HJB equation

$$\frac{\partial W}{\partial t} = \max_w \{ - L_f W(t, x, w) - L_g W(t, x, w)u + |h(x)|^2 \tag{36}$$
$$+ |u|^2 - \gamma^2 |w|^2 - \gamma^2 |y - c(x)|^2 \} \,,$$

with the boundary condition $N(x)$ at $t = 0$.

The significance of the dual concepts of *cost-to-go* and *cost-to-come* functions is that for any time-state pair (t, x) they allow the total cost to be additively decomposed into two parts: forward-looking and backward-looking. A further maximization over x at the instant when they meet yields a performance-driven worst value of the state at that instant, as a function of the current and past values of the measurement y, that is,

$$\hat{x}(t) = \arg \max_x [V(x) + W(t, x)] \,, \tag{37}$$

where the dependence on y comes through the cost-to-come function W. If the maximum is unique, then *certainty equivalence* applies, which means that

a control that guarantees a disturbance attenuation level of γ is the solution $u^\star(\hat{x}(t))$ of the state feedback problem and is obtained from the HJI equation

$$\min_u\{\max_w[L_fV(x,w) + L_gV(x,w)u + |h(x)|^2 + |u|^2 - \gamma^2|w|^2]\} = 0\,. \quad (38)$$

The task of finding the cost-to-come function, studied by Helton and James (1999) and others, is extremely difficult. The dependence on the measurement history makes solving the forward equation (36) generally an infinite dimensional problem. Only in problems with special structures has it been possible to obtain finite dimensional solutions, as in the linear-quadratic problem (the $\mathcal{H}_\infty$ control problem). In this case, W is a quadratic function of x, and depends on u and y linearly. A finite dimensional solution can also be obtained for the class of worst case parameter identification problems where the system dynamics are nonlinear, but the unknown constant parameters enter linearly. In this case the cost-to-come analysis of Didinsky *et al.* (1995) leads to explicit expressions for a class of robust identifiers, with a built-in disturbance attenuation feature. Another class of problems where the cost-to-come function can be computed explicitly (and is finite dimensional) is adaptive control (formulated as disturbance attenuation) where the system is in strict feedback form, and the unknown parameters again enter linearly. Although in this case the maximum in (37) is not unique, explicit constructions for the disturbance attenuating controllers were obtained with state feedback by Pan and Başar (1998), and with output feedback by Tezcan and Başar (1999).

3.6   *Nonlinear Relative Degree and Zero Dynamics*

The development of nonlinear geometric methods was a remarkable achievement of the 1980's, presented in the books by Isidori (1995), Nijmeijer and van der Schaft (1990), Marino and Tomei (1995) and in the numerous papers referenced therein. Geometric concepts permeate our current thinking about nonlinear systems. Two of them need to be made explicit here: nonlinear *relative degree* and *zero dynamics*. These indispensable tools bring into focus the common input-output structure of linear and nonlinear systems.

For a scalar transfer function, the relative degree is the difference between the number of poles and zeros. This is also the number of times the output $y(t)$ needs to be differentiated for the input $u(t)$ to appear. For a state-space realization (A,b,c,d), the relative degree is zero if $d \neq 0$, it is one if $d = 0$ and $cb \neq 0$, it is two if $d = 0$, $cb = 0$ and $cAb \neq 0$, etc. For the nonlinear system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x) + j(x)u\,, \quad x \in \mathbb{R}^n, u, y \in \mathbb{R}\,,\end{aligned} \quad (39)$$

the relative degree at a point $x^\star$ is zero if $j(x^\star) \neq 0$, it is one if $j(x^\star)$ is identically zero on a neighborhood of $x^\star$ and $L_g h \neq 0$ at $x^\star$. This is so because

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = L_f h + L_g h \, u \,, \tag{40}$$

so that, if $L_g h$ is nonzero, then the input $u(t)$ appears in the expression for the first derivative $\dot{y}(t)$ of the output $y(t)$. If $L_g h$ is zero, we can differentiate $\dot{y}$ once more and check whether u appears in the expression for $\ddot{y}(t)$, etc. In contrast to linear systems, the relative degree of nonlinear systems may not be defined.

When the system (39) has relative degree one, its *input-output linearization* is performed with the feedback transformation

$$u = (L_g h)^{-1}(v - L_f h) \quad \Rightarrow \quad \dot{y} = v \,, \tag{41}$$

which cancels the nonlinearities in the $\dot{y}$-equation and converts it into $\dot{y} = v$. Selecting new state coordinates in which y is one of the states, the remaining $n - 1$ equations with $y(t) \equiv 0$ and $v(t) \equiv 0$ constitute the *zero dynamics*, that is, the dynamics which remain when the output is kept at zero. If the relative degree is two, then the linear part of the system is $\ddot{y} = v$, the chain of two integrators. In this case the zero dynamics are described by the remaining $n - 2$ equations $y(t) = \dot{y}(t) \equiv 0$ and $v(t) \equiv 0$.

The relative degree and the zero dynamics cannot be altered by feedback. For this reason, systems with unstable zero dynamics, *nonminimum phase systems*, are much harder to control than *minimum phase systems* in which the zero dynamics are asymptotically stable. In *weakly minimum phase systems* the zero dynamics are stable, but not asymptotically stable.

Two *caveats* need to be made about input-output linearization (41) as a design tool. First, there may be nonlinearities that should not be canceled because they help the design task, like $-x^3$ which helps us to stabilize $\dot{x} = x - x^3 + u$. Second, in the presence of modeling errors, the concepts of relative degree and zero dynamics may be nonrobust. Sastry *et al.* (1989) showed that regular perturbations in a system may lead to singularly perturbed unstable zero dynamics. It is therefore important that geometric concepts be applied jointly with the analytical tools needed to guarantee robustness.

17

3.7 Feedback Passivation

Achieving strict passivity (SPR) with feedback was, in the 70's, a common tool for adaptive control of linear systems. A result of Fradkov (1976), made more accessible by Fradkov and Hill (1998), is that (A, B, C) can be rendered SPR with feedback if and only if it is minimum phase and relative degree one. In nonlinear control, the use of passivation was motivated by a difficulty encountered in feedback stabilization of linear-nonlinear cascade systems

$$\begin{aligned} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= A\xi + Bu \end{aligned} \tag{42}$$

resulting from input-output linearization. The difficulty was that the GAS property of the subsystem $\dot{x} = f(x, 0)$ is not sufficient to achieve GAS of the whole cascade with ξ-feedback $u = K\xi$, as illustrated by

$$\begin{aligned} \dot{x} &= -x + x^2\xi \\ \dot{\xi} &= u \, . \end{aligned} \tag{43}$$

With feedback $u = k\xi$, for every finite $k < 0$, there exist initial conditions from which $x(t)$ escapes to infinity. Thus, feedback is required from both ξ and x, that is,

$$u = K\xi + v(x, \xi) \, . \tag{44}$$

Such a control law was designed by Byrnes and Isidori (1989) for the special case of (42) with $\dot{\xi} = Bu$, where B is a square nonsingular matrix. Kokotović and Sussmann (1989) extended this design to *feedback passivation* where the cascade (42) is represented as the feedback interconnection of the blocks H_1 and H_2 in Figure 1. The final result in Figure 3 is arrived at in several steps. First, an output η of the linear block H_1 is selected to be the input of the nonlinear block H_2, that is, the x-subsystem of (42) is rewritten as

$$\dot{x} = f(x, 0) + g(x, \xi)\eta \, , \tag{45}$$

where several choices of $\eta = C\xi$ may be available. An output y is then chosen to render (45) passive from η to y. If a Lyapunov function $V(x)$ is known for $\dot{x} = f(x, 0)$ so that $L_fV \leq 0$, then $y = L_gV^T$ renders (45) passive because

$$\dot{V} = L_fV + L_gV\eta \leq L_gV\eta = y^T\eta \, . \tag{46}$$

18

Finally, if the linear block H_1 is rendered PR by feedback $K\xi$, the passivity theorem will be satisfied by closing the loop with $-y = -L_g V^T$ as in Figure 3.
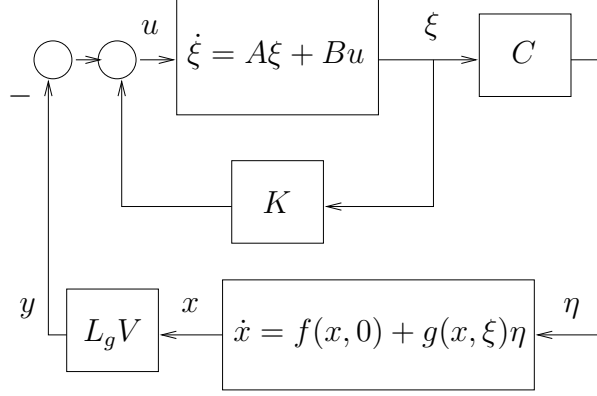

Fig. 3. Feedback passivation design.

For the existence of K in the global stabilization of the linear-nonlinear cascade (42) with (44), Kokotović and Sussmann (1989), and Saberi *et al.* (1990) showed that the weak minimum phase property of (A, B, C) is necessary unless some other restriction is imposed on the nonlinear part. Upon an extension by Ortega (1989), Byrnes *et al.* (1991) proceeded to prove that at $x = 0$, the nonlinear system (39) with $j(x) \equiv 0$ is *feedback passive* with a positive definite storage function $S(x)$ if and only if it is *relative degree one* and *weakly minimum phase*. Indeed, when the condition $(L_g S)^T(x) = h(x)$ of the nonlinear PR Lemma (5) is differentiated, noting that $\frac{\partial S}{\partial x} = 0$ at $x = 0$, the result is

$$g^T \frac{\partial^2 S}{\partial x^2} g = L_g h \quad \text{at } x = 0.$$

Along with $rank\left\{\frac{\partial h}{\partial x}(0)\right\} = m$, this implies that the relative degree is one. To deduce the weak minimum phase condition we differentiate $(L_g S)^T(x) = h(x)$ with respect to time in the zero dynamics manifold $h(x) \equiv 0$. Then we ascertain from $\dot{S} \leq uy$ and $y(t) \equiv 0$ that $L_f S \leq 0$, which is the weak minimum phase property.

An in-depth study of obstacles to global, or even semiglobal[3] stabilization of the cascade (42) was initiated by Sussmann (1990), and pursued by Sussmann and Kokotović (1991), and Byrnes and Isidori (1991). One of the main obstacles was identified to be the *peaking phenomenon* caused by high-gain feedback $u = K\xi$. A further analysis by Sepulchre *et al.* (1997) and Sepulchre (2000)

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[3] The term *semiglobal stabilizability* means that for any desired finite region of attraction, a feedback controller exists.

showed that higher relative degree systems are prone to destabilizing transients caused by not only fast but also slow peaking. For nonminimum phase systems global stabilization can be achieved only with further restrictions on the cross-term $g(x, \xi)$, as discussed by Braslavsky and Middleton (1996), and Sepulchre and Arcak (1998), where these restrictions are characterized by a relationship between the locations of the nonminimum phase zeros and the growth of $g(x, \xi)$ in x and ξ.

3.8 Stability Margins

Small-gain and passivation designs guarantee nonlinear analogs of gain and phase margins for several types of dynamic uncertainties, as in the system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)[u + w(x, z, u)] \\
\dot{z} &= q(x, z, u),
\end{aligned}
\tag{47}
$$

where the z-subsystem with the output $y = u + w(x, z, u)$ represents *unmodeled dynamics*. A GAS control law $\alpha(x)$ designed for the nominal model $\dot{x} = f(x) + g(x)u$, will in general fail to achieve GAS of the actual system (47). Small-gain redesigns applying condition (15) were proposed by Jiang *et al.* (1994), Krstić *et al.* (1996), Praly and Wang (1996), and Jiang and Mareels (1997). As an illustration we let $w(x, z, u) = z$, $q(x, z, u) = q(z, x)$ and assume that the unmodeled dynamics are ISS with x considered as input, that is,

$$
|z(t)| \leq \max \left\{ \beta(|z(0)|, t), \gamma_1 \left(\sup_{0 \leq \tau \leq t} |x(\tau)| \right) \right\}.
\tag{48}
$$

The nominal control law $\alpha(x)$ was designed for $V(x)$, such that $L_{f+g\alpha} V < 0$, $x \neq 0$. For redesign, we select a class-$\mathcal{K}$ function $\gamma_2(\cdot)$ such that $\gamma_1 \circ \gamma_2(s) < s$, to be assigned as the ISS-gain from w to x. This gain assignment is achieved by a continuous approximation of the redesigned control law

$$
u = \alpha(x) - \operatorname{sgn}(L_g V(x)) \rho^{-1}(|x|),
\tag{49}
$$

where $\rho(\cdot)$ is determined from $\gamma_2(s) = \sigma_1^{-1} \circ \sigma_2 \circ \rho(s)$, with $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ as in (11). The resulting feedback system can tolerate all unmodeled dynamics that satisfy (48). In this sense, (48) represents an ISS-*gain margin.*

An alternative redesign by passivation does not require that unmodeled dynamics have bounded ISS-gain. Instead, the class of unmodeled dynamics is restricted by a passivity requirement on the z-subsystem in (47) with u as the input and $y = u + w(x, z, u)$ as the output.

The passivation redesigns of Janković *et al.* (1999b), extended by Hamzi and Praly (1999), are based on $V(x)$ as a control Lyapunov function (CLF) for the nominal system $\dot{x} = f(x) + g(x)u$. For example, if $V(x)$ has the property

$$L_f V(x) < |L_g V(x)|^2, \quad \forall x \neq 0, \tag{50}$$

then the control law

$$u = -k L_g V(x), \quad k \geq 1 \tag{51}$$

guarantees GAS not only for the nominal system, but also for all stable un-modeled dynamics which remain passive with the output $y - \frac{1}{k}u$. This stability margin is due to the fact that the control law in (51) is optimal with respect to (6) with $R(x) = I$, because then the value function $V(x)$ satisfies (50). For the case when $V(x)$ does not satisfy (50), Janković *et al.* (1999b) construct a new $\tilde{V}(x)$ which recovers the same margin.

Both small-gain and passivity margins restrict the unmodeled dynamics to have relative degree zero. With a higher relative degree, the preserved prop-erties may not be global. A singular perturbation result (Sepulchre *et al.*, 1997, Theorem 3.18) shows that they can be preserved in large regions if the unmodeled dynamics are much faster than the nominal closed loop system. For *feedforward systems*, the redesign by Arcak *et al.* (2000) achieves global robustness for a wide range of unmodeled dynamics.

4 DESIGN PROCEDURES

For nonlinear control the 1990's started with a breakthrough: *backstepping*, a recursive design for systems with nonlinearities not constrained by linear bounds. Although the idea of integrator backstepping may be implicit in some earlier works, its use as a design tool was initiated by Tsinias (1989b, 1991), Byrnes and Isidori (1989), Sontag and Sussmann (1988), Kokotović and Sussmann (1989), and Saberi *et al.* (1990). However, the true potential of backstepping was discovered only when this approach was developed for nonlinear systems with structured uncertainty. With *adaptive backstepping*, Kanellakopoulos *et al.* (1991a,b) achieved global stabilization in the presence of unknown parameters, and with *robust backstepping*, Freeman and Koko-tović (1992, 1993), and Marino and Tomei (1993b) achieved it in the presence of disturbances. The emergence of adaptive, robust and observer-based back-stepping was described in the 1991 Bode Lecture, Kokotović (1992).

The ease with which backstepping incorporated uncertainties and unknown

parameters contributed to its instant popularity and rapid acceptance. At the same time, its limitation to a class of pure feedback (lower triangular) systems stimulated the development of other recursive procedures, such as *forwarding* by Teel (1992), Mazenc and Praly (1996), and Janković *et al.* (1996), applicable to feedforward systems. Interlacing the steps of these procedures, it is often possible to design other types of systems. The rapidly growing literature on recursive nonlinear designs includes the books by Krstić *et al.* (1995), Marino and Tomei (1995), Freeman and Kokotović (1996*b*), Sepulchre *et al.* (1997), Krstić and Deng (1998), Dawson *et al.* (1998), and Isidori (1999).

4.1   Construction of RCLF's by Backstepping

The purpose of backstepping is the construction of various types of CLF's: robust, adaptive etc. Backstepping constructions of RCLF's by Freeman and Kokotović (1992), and Marino and Tomei (1993*b*) are illustrated on the system

$$\dot{x}_1 = x_2 + w_1(x, t) \tag{52}$$
$$\dot{x}_2 = u + w_2(x, t) \,,$$

where the uncertainties w_1 and w_2 are bounded by known functions

$$|w_1(x, t)| \leq \Delta_1(x_1) \tag{53}$$
$$|w_2(x, t)| \leq \Delta_2(x_1, x_2) \,,$$

which are allowed to grow faster than linear, like $\Delta_1(x_1) = x_1^2$. The crucial restriction of backstepping is imposed on the structure of bounding functions Δ_1, Δ_2 in (53), allowing Δ_i to depend only on $x_1, \cdots, x_i$. For the ease of presentation it will be assumed that $\Delta_1(0) = 0$, $\Delta_2(0, 0) = 0$, and that the derivative of $\Delta_1(x_1)$ exists and is zero at $x_1 = 0$. When this is not the case, a slightly modified procedure achieves boundedness and convergence to a compact set around $x = 0$.

Backstepping starts with a part of the system for which the construction of an RCLF is easy, as in the case when the uncertainty is *matched*. Lyapunov minmax designs for matched uncertainties were developed around 1980 by Gutman (1979), Corless and Leitmann (1981) and others, presented in (Khalil, 1996*b*, Section 13.1).

In the first equation of (52) the uncertainty w_1 is *matched* with x_2. This means that if x_2 were our control, it would be able to counteract the worst case of w_1 by $x_2 = \mu_1(x_1)$. To design such a *virtual* control law $\mu_1(x_1)$ for the x_1-equation we can use $V_1 = x_1^2$ as our RCLF. Then to render $\dot{V}_1$ negative we seek $\mu_1(x_1)$

which, for $x_1 \neq 0$ and all $w_1(x, t)$ bounded by (53), satisfies

$$x_1[\mu_1(x_1) + w_1(x, t)] \leq x_1\mu_1(x_1) + |x_1|\Delta_1(x_1) < 0 . \tag{54}$$

A possible choice is

$$\mu_1(x_1) = -x_1 - \operatorname{sgn}(x_1)\Delta_1(x_1) , \tag{55}$$

where $\mu_1'(x_1) := d\mu_1/dx_1$ exists because of the assumptions on Δ_1.

It is consistent with the idea of x_2 being a *virtual control* that we think of $x_2 - \mu_1(x_1)$ as an *error* to be regulated to zero by the actual control u. This suggests that we examine

$$V_2(x) = V_1(x_1) + [x_2 - \mu_1(x_1)]^2 \tag{56}$$

as a candidate RCLF for the whole system (52). Our task is then to achieve, with some $u = \mu_2(x)$,

$$\dot{V}_2 = 2x_1[x_2 + w_1] + 2[x_2 - \mu_1(x_1)][u + w_2 - \mu_1'(x_1)(x_2 + w_1)] < 0 \tag{57}$$

for all $x \neq 0$, and all admissible $w_1(x, t)$ and $w_2(x, t)$. The choice of $\mu_1(x_1)$ in (55) to satisfy (54) has made this task easy, because it has reduced (57) to

$$\dot{V}_2 \leq -2x_1^2 + 2[x_2 - \mu_1(x_1)] [x_1 + u + w_2 - \mu_1'(x_1)(x_2 + w_1)] < 0 , \tag{58}$$

where u matches the composite uncertainty

$$w_c(x, t) := w_2(x, t) - \mu_1'(x_1)w_1(x, t) , \tag{59}$$

with the bound $|w_c(x, t)| < \Delta_c(x)$ computed from Δ_1, Δ_2 and μ_1'. We first let

$$u = \mu_2(x) = -[x_2 - \mu_1(x)] - x_1 + \mu_1'(x_1)x_2 + u_r(x) . \tag{60}$$

Then, the inequality to be satisfied by $u_r(x)$ is of the same form as the inequality (54) and, hence,

$$u_r(x) = -\operatorname{sgn}[x_2 - \mu_1(x_1)]\Delta_c(x) . \tag{61}$$

The so designed $\mu_2(x)$ yields

$$\dot{V}_2 \leq -2x_1^2 - 2[x_2 - \mu_1(x_1)]^2 , \tag{62}$$

23

which means that GAS is achieved.

This example highlights the key recursive feature of backstepping: the RCLF for step $k+1$ is constructed as

$$V_{k+1} = V_k + [x_k - \mu_{k-1}(x_1, \cdots, x_{k-1})]^2, \tag{63}$$

where V_k is the k-th RCLF and μ_{k-1} is the virtual control law which renders $\dot{V}_k < 0$ for $x_k = \mu_{k-1}(x_1, \cdots, x_{k-1})$.

Backstepping also serves for ISS-CLF construction, developed by Praly and Jiang (1993), Jiang et al. (1994), Krstić et al. (1995), and illustrated here on the system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi + p(x)w \\
\dot{\xi} &= u \,.
\end{aligned}
\tag{64}
$$

This is the system (20) augmented by one integrator. We assume that $V_1(x)$ is an ISS-CLF for the x-subsystem with ξ as its virtual control. In other words, we can find $\mu_1(x)$ such that $\xi = \mu_1(x)$ satisfies the dissipation inequality

$$\dot{V}_1 = L_{f+g\mu_1}V_1 + L_pV_1 w \le -\alpha_1(|x|) + \beta_1(|w|), \tag{65}$$

with a class-$\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ and a class-$\mathcal{K}$ function $\beta_1(\cdot)$. Then, an ISS-CLF for (64) is

$$V_2(x, \xi) = V_1(x) + [\xi - \mu_1(x)]^2, \tag{66}$$

and, with the control law

$$u = \mu_2(x, \xi) = -(1 + |L_p\mu_1|^2)(\xi - \mu_1) - L_gV_1 + L_{f+g\xi}\mu_1, \tag{67}$$

the closed-loop system (64)-(67) has the ISS-property

$$\dot{V}_2 \le -\alpha_2\left(\left\|\begin{matrix} x \\ \xi \end{matrix}\right\|\right) + \beta_2(|w|), \tag{68}$$

which is analogous to the ISS property (65).

Teel and Praly (2000) considered the problem of assigning a general supply rate $\alpha(x, \xi, w)$ instead of $-\alpha_2 + \beta_2$ in (68). Backstepping procedures for ISS, $\mathcal{L}_2$, and similar gain assignment tasks appear as special cases of their procedure.

Marino *et al.* (1994), and Isidori (1996*b,a*) employed backstepping to solve an *almost disturbance decoupling* problem. For systems with stochastic disturbances backstepping designs were developed by Krstić and Deng (1998), and Pan and Başar (1999). Freeman and Praly (1998) extended backstepping to control inputs with magnitude and rate limits, and Jiang and Nijmeijer (1997) to nonholonomic systems. An undesirable property of backstepping is the growth of 'nonlinear gains', which Freeman and Kokotović (1993) counteracted by 'flattened' Lyapunov functions.

4.2   Backstepping with Optimality

With backstepping we can construct RCLF's, ISS-CLF's or NSS-CLF's for systems in the *strict feedback form*

$$\dot{x}_i = f_i(x^i) + g_i(x^i)x_{i+1} + p_i(x^i)^T w$$
$$\dot{x}_n = f_n(x) + g_n(x)u + p_n(x)^T w \,, \tag{69}$$

where $x^i := (x_1, \cdots, x_i)^T$, $x = x^n$, and $g_i \neq 0$, $i = 1, \cdots, n$, for all x. With an ISS-CLF obtained at the last step, we can design an ISS control law. It is of practical interest to render this design inverse optimal, that is, to verify that the constructed ISS-CLF satisfies an Isaacs inequality. Several inverse optimal constructions were proposed by Pan and Başar (1998), Krstić and Deng (1998) and Ezal *et al.* (2000). The construction by Ezal *et al.* (2000) is particularly useful because it also achieves *local optimality*, that is, the linearization of the designed nonlinear feedback system is $\mathcal{H}_\infty$-optimal. In this way earlier optimal designs for linear systems are incorporated in nonlinear designs.

The *locally optimal backstepping* is now illustrated on the system

$$\dot{x}_1 = x_1^2 + x_2 + w \tag{70}$$
$$\dot{x}_2 = u \,,$$

with the prescribed local cost

$$J = \int\limits_0^\infty (x_1^2 + x_2^2 + u^2 - \gamma^2 w^2)dt \,. \tag{71}$$

When the nonlinearity x_1^2 is ignored, this is a linear $\mathcal{H}_\infty$ problem with full state measurement. For the linear problem the limiting attenuation level is $\gamma^\star = 1.27$, so we select $\gamma = 5$ as the desired level. The $\mathcal{H}_\infty$-optimal linear control $u_{lin} = -1.06x_1 - 1.78x_2$ is easily calculated via the Riccati matrix P.

To retain u_{lin} as the linear part of the nonlinear backstepping control law, Ezal *et al.* (2000) used the Cholesky factorization $P = L^T DL$, where D is diagonal and L is lower triangular with the identity in its diagonal. The rest of the nonzero entries of L serve as coefficients, row by row, for the linear parts of virtual control laws. The derivative of $V_1 = 1.18x_1^2$, where 1.18 comes from D, is expressed as

$$\dot{V}_1 = -1.36x_1^2 + 25w^2 - 25(w - \nu_1)^2 + 2.36x_1(x_2 - \mu_1), \qquad (72)$$

where $\nu_1 = \frac{1}{\gamma^2}1.18x_1 = 0.05x_1$ is the worst case disturbance. The virtual control that renders $\dot{V}$ most negative for $w = \nu_1$ is $\mu_1(x_1) = -0.6x_1 - x_1^2$, where -0.6 comes from L and $-x_1^2$ cancels the nonlinearity[4]. This $\mu_1(x_1)$ satisfies the dissipation inequality

$$\dot{V}_1 \leq -1.36x_1^2 + 25w^2. \qquad (73)$$

The final ISS-CLF is $V_2 = V_1 + 1.78(x_2 - \mu_1(x_1))^2$, where 1.78 comes from D.

For the worst case disturbance the optimal control is

$$u = -1.78\, r^{-1}(x)(x_2 - \mu_1(x_1)). \qquad (74)$$

Meaningful penalties $q(x)$ and $r(x)$ for inverse optimality are obtained with

$$r^{-1}(x) = \begin{cases} 1 + \sigma(x) & \text{if } \sigma(x) \geq 0 \\ 1 & \text{if } \sigma(x) < 0. \end{cases} \qquad (75)$$

A possible choice, $\sigma(x) = 1.8x_1 + 1.05(x_2 - \mu_1(x_1))^2$, renders $q(x)$ positive definite. Other choices can be made, but, to be consistent with the local $\mathcal{H}_\infty$-optimal problem, they all must satisfy $r(0) = R$ and $\frac{\partial^2 q}{\partial x^2}(0) = Q$, where Q and R are the penalty matrices in the prescribed quadratic $\mathcal{H}_\infty$-cost. In the above example $Q = I$, $R = 1$.

The superiority of the nonlinear design is visible from the solutions plotted in Figure 4 for the case when $w = 0$. With the linear control law, the stability region is only to the left of the boundary M_s, while with the nonlinear control it is the whole plane. The nonlinear controller not only achieves GAS but it also improves the overall performance. This can be seen from the pair of trajectories marked by A, where the transient swing of the solid curve is much smaller.

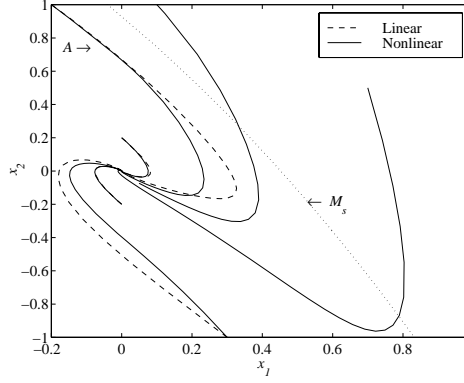[4] In virtual control laws cancelation is harmless, but is to be avoided in the actual control law.

Fig. 4. Linear (dashed), and nonlinear (solid) designs.

4.3 Adaptive Nonlinear Control

In the adaptive control problem the uncertainty is an unknown parameter vector θ and its estimate $\hat{\theta}(t)$ is used in the design of a control law. A *certainty equivalence* design, common in adaptive linear control, is not applicable to systems with strong nonlinearities like x^2. To see why, consider the system

$$\dot{x} = x + \theta x^2 + u \,, \tag{76}$$

and let its certainty equivalence control be $u = -2x - \hat{\theta}x^2$. It turns out that even with an exponentially convergent estimate $|\hat{\theta}(t) - \theta| \leq ce^{-at}$, some solutions of

$$\dot{x} = -x - (\hat{\theta} - \theta)x^2 \tag{77}$$

escape to infinity. For the matched case (76), the standard Lyapunov design furnishes a parameter update law which is faster than exponential. This design was extended by Kanellakopoulos *et al.* (1991*c*) to systems in which θ is separated from u by no more than one integrator, like $\dot{x}_1 = x_2 + \theta x_1^2$; $\dot{x}_2 = u$.

The real difficulties were encountered in the 'benchmark problem'

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta x_1^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \,,
\end{aligned} \tag{78}$$

presented by Kokotović and Kanellakopoulos (1990). Global stabilization of (78), and convergence of $x(t)$ were finally achieved with the first, overparametrized version of adaptive backstepping by Kanellakopoulos *et al.* (1991*a,b*), which also employed the *nonlinear damping* of Feuer and Morse (1978). Jiang and

27

Praly (1991) reduced the overparametrization by one half, and the *tuning functions* method of Krstić *et al.* (1992) completely removed it.

The current form of adaptive backstepping, described in the book by Krstić *et al.* (1995), will now be explained with the help of the *adaptive* CLF (ACLF). For the x-subsystem of the augmented system

$$\dot{x} = f(x) + F(x)\theta + g(x)\xi \tag{79}$$
$$\dot{\xi} = u, \qquad x \in \mathbb{R}^n; \ \xi, u \in \mathbb{R}, \ \theta \in \mathbb{R}^p, \tag{80}$$

with ξ as its virtual control, $V(x, \theta)$ is an ACLF if there exists $\alpha_1(x, \theta)$ such that for all $x \neq 0$, and all θ,

$$\frac{\partial V_1}{\partial x_1} \left[f(x) + F(x) \left(\theta + \frac{\partial V_1}{\partial \theta}^T \right) + g(x)\alpha_1(x, \theta) \right] < -\sigma_1(x, \theta), \tag{81}$$

where $\sigma_1(x, \theta) \geq 0$. Then, a virtual adaptive controller for the x-subsystem is

$$\xi = \alpha_1(x, \hat{\theta}) \tag{82}$$
$$\dot{\hat{\theta}} = \tau_1(x, \hat{\theta}) := F^T(x)\frac{\partial V_1}{\partial x_1}^T (x, \hat{\theta}),$$

where τ_1 is the *first tuning function*. The stability properties of the feedback system (79),(82) are established with

$$\bar{V}_1(x, \hat{\theta}) = V_1(x, \hat{\theta}) + \frac{1}{2}|\hat{\theta} - \theta|^2. \tag{83}$$

As always, the purpose of backstepping is to construct a CLF, in this case an ACLF, for the augmented system (79),(80). Again, a candidate is

$$V_2(x, \xi, \theta) = V_1(x, \theta) + \frac{1}{2}(\xi - \alpha_1(x, \theta))^2. \tag{84}$$

This candidate wins, because there exists $\alpha_2(x, \xi, \theta)$ and $\sigma_2(x, \xi, \theta) \geq 0$ such that

$$\frac{\partial V_2}{\partial(x, \xi)} \begin{bmatrix} f(x) + F(x)\left(\theta + \frac{\partial V_2}{\partial \theta}^T\right) + g(x)\xi \\ \alpha_2(x, \xi, \theta) \end{bmatrix} < -\sigma_2(x, \xi, \theta), \tag{85}$$

for all $x \neq 0$, $\xi \neq 0$, where expressions for $\alpha_2(x, \xi, \theta)$ and $\sigma_2(x, \xi, \theta)$ can be obtained by a short calculation. With $V_2(x, \xi, \theta)$ as an ACLF, an adaptive

28

controller for (79), (80) is

$$u = \alpha_2(x, \xi, \hat{\theta}), \qquad \dot{\hat{\theta}} = \tau_2(x, \xi, \hat{\theta}), \tag{86}$$

where the update law is the *second tuning function*

$$\tau_2(x, \xi, \hat{\theta}) = \tau_1(x, \hat{\theta}) - \left(\frac{\partial \alpha_1}{\partial x}\right)^T (\xi - \alpha_1). \tag{87}$$

The boundedness of $x(t)$, $\xi(t)$, $\hat{\theta}(t)$ and the convergence $x(t) \to 0$, $\xi(t) \to 0$ are easy to prove with

$$\bar{V}_2 = V_1(x, \xi, \hat{\theta}) + \frac{1}{2}|\hat{\theta} - \theta|^2. \tag{88}$$

The recursive formula for ACLF's V_i is as in (84) and for the tuning functions τ_i is as in (87). A similar recursive formula is available for α_i.

An alternative estimation-based approach to adaptive nonlinear control was motivated by adaptive designs for linear systems. The status of this line of research in 1990 was described by Praly *et al.* (1991). For an estimation-based design to succeed in nonlinear systems, the traditional *certainty equivalence* control law had to be replaced by a stronger control law. Krstić and Kokotović (1995, 1996) used ISS-backstepping to achieve ISS properties with respect to $\hat{\theta}(t) - \theta$ and its derivative as unknown bounded disturbances. This ISS controller can be used in conjunction with most standard adaptive estimators.

Because the newly developed adaptive nonlinear controllers had no counterparts in adaptive linear control, it was of interest to specialize them to linear systems and compare them with traditional adaptive controllers. Krstić *et al.* (1994) showed that the new designs far outperformed their predecessors.

Extensions of adaptive backstepping to a wider class of systems were made by Seto *et al.* (1994). Asymptotic properties, transient performance, robustness and dynamic extensions of the new adaptive controllers were further investigated by Zhang *et al.* (1996), Ikhouane and Krstić (1998), Lin and Kanellakopoulos (1998), Sira-Ramírez *et al.* (1997), Jiang and Praly (1998) and several other authors. The systems in the form (69) containing both unknown parameters θ and bounded disturbances $w(x, t)$ can be handled by a combination of adaptive and robust backstepping as described by Freeman *et al.* (1998*b*). The difficult problem of nonlinear parameterizations has recently been addressed by Bošković (1998), Annaswamy *et al.* (1998), and Kojić *et al.* (1998).

4.4 Nested Saturation and Forwarding

Backstepping does not apply to systems with feedforward paths, such as

$$
\begin{aligned}
\dot{x}_1 &= x_2 + x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \,,
\end{aligned}
\tag{89}
$$

where x_3^2 in the first equation constitutes a path bypassing the x_2-integrator. With his *nested saturation* procedure, Teel (1992, 1996a) initiated the development of a family of *forwarding* designs applicable to *feedforward systems*, that is, systems without feedback paths like

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(x_2, x_3, u) \\
\dot{x}_2 &= x_3 + \varphi_2(x_3, u) \\
\dot{x}_3 &= u \,.
\end{aligned}
\tag{90}
$$

The only open-loop instability in these systems is due to the chain of integrators which is easy to stabilize with linear feedback $u = Kx$. However, this may result in an insufficient stability region because of the destabilizing feedback loops closed through feedforward nonlinearities like x_3^2 in (89). To keep the gains of these loops small for large x, Teel employed saturation elements, nested loop by loop.

For the benchmark system (89), Teel started by stabilizing the (x_2, x_3)-subsystem with a linear feedback, say, $u = -x_2 - x_3 + v$, where v is a new control variable. Then, using $z := x_1 + x_2 + x_3$ to replace x_1 in (89) yields

$$
\begin{aligned}
\dot{z} &= x_3^2 + v \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -x_2 - x_3 + v \,.
\end{aligned}
\tag{91}
$$

At this point a saturation element $v = -\phi(z)$ is employed to guarantee that the feedback interconnection in Figure 5 of the (x_2, x_3)-block $H_1(s) = \frac{s}{s^2+s+1}$ and the nonlinear z-block H_2 satisfies a small-gain condition as in Teel (1996a). For system (90) one more saturation element may be needed for gain reduction in the (x_2, x_3)-subsystem because of its nonlinearity $\varphi_2(x_3, u)$.

The nested saturation procedure was extended by Teel (1996a) to a general class of feedforward systems including the systems considered by Sussmann *et al.* (1994).
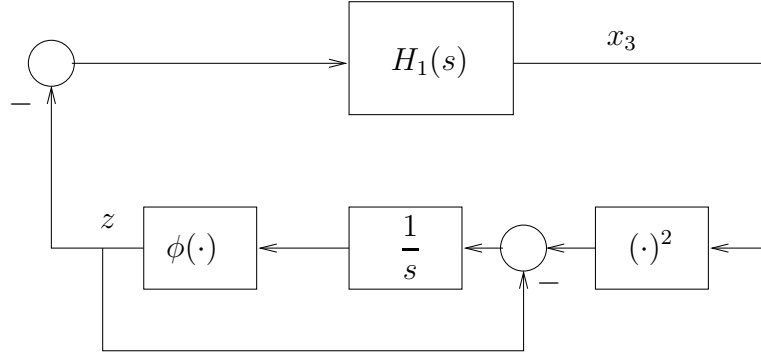
Fig. 5. Achieving small-gain with a saturation element.

An alternative to Teel's procedure is the Lyapunov *forwarding procedure* developed by Mazenc and Praly (1996), and Janković *et al.* (1996). It treats a feedforward system as a connection of cascade subsystems in the form

$$
\begin{aligned}
\dot{z} &= f(z) + \psi(z,\xi) \\
\dot{\xi} &= a(\xi) \,,
\end{aligned} \tag{92}
$$

where $\dot{\xi} = a(\xi)$ with a Lyapunov function $U(\xi)$ is GAS and locally exponentially stable. The growth of $|\psi(z,\xi)|$ in $|z|$ is not higher than linear. The subsystem $\dot{z} = f(z)$ is globally stable with a Lyapunov function $W(z)$, that is, $L_f W(z) \leq 0$ for all z. For the cascade (92) a Lyapunov function $V_0(z,\xi)$ constructed by Janković *et al.* (1996) is of the form

$$
V_0(z,\xi) = W(z) + \Psi(z,\xi) + U(\xi) \,, \tag{93}
$$

where $W(z)$ for $\dot{z} = f(z)$ and $U(\xi)$ for $\dot{\xi} = a(\xi)$ are known, and the cross term Ψ is to be constructed to satisfy $\dot{\Psi} = -L_\psi W$, so that

$$
\dot{V}_0 = L_f W + L_a U \leq 0 \,. \tag{94}
$$

The main burden of forwarding by Janković *et al.* (1996) is the evaluation of the integral

$$
\Psi(z,\xi) = \int_0^\infty L_\psi W(\tilde{z}(t,z,\xi),\tilde{\xi}(t,\xi))dt \,, \tag{95}
$$

along the solutions of $\tilde{z}(t,z,\xi)$, $\tilde{\xi}(t,\xi)$ of (92) starting from (z,ξ) at $t = 0$. In many cases this requires numerical integration, but there are problems when $\Psi(z,\xi)$ can be obtained in closed form, like when $f(z)$ and $a(\xi)$ are linear and $\psi(z,\xi) = p(\xi)$ is a polynomial. When (92) has an invariant manifold decomposition, Mazenc and Praly (1996) do not employ the cross-term Ψ.

Instead of W and U, they introduce the 'nonlinear scaling' $\ell(W)$ and $\rho(U)$ and a change of coordinates.

Recent extensions to forwarding designs were presented by Mazenc (1997), Grognard *et al.* (1999), Lin and Qian (1998), and Arcak *et al.* (2000).

4.5   Interlacing and Indirect Passivation

When none of the recursive procedures is individually applicable to a system, their 'interlaced' application may lead to a constructive design, as in (Sepulchre *et al.*, 1997, Section 6.3). For example, a stabilizing control law for the system

$$
\begin{aligned}
\dot{x}_1 &= x_1 + x_2 + x_3^3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 + u \,,
\end{aligned}
\tag{96}
$$

can be designed using $\tilde{x}_2 = x_1 + x_2$ instead of x_2 and then performing one step of forwarding followed by one step of backstepping.

The system (96) will now be used to illustrate the *indirect passivation* design of Larsen and Kokotović (1998), and Janković *et al.* (1999*a*). The goal is to render the linear part of (96) passive from $v = -x_3^3$ to $y = x_3$, and then establish GAS with the passivity theorem and detectability. Because the relative degree from v to y is two, the control law $u = Kx + \beta v$ is employed to lower the relative degree to one. The next task is to find K and β to satisfy the PR property from v to y. This is achieved using LMI's, and the resulting control law

$$
u = k_1 x_1 + k_2 x_2 + k_3 x_3 + \beta x_3^3 \,,
\tag{97}
$$

achieves GAS by the passivity theorem. The rich literature on applications of LMI's to control problems is summarized in the book by Boyd *et al.* (1994).

4.6   Output Feedback Designs

Progress in nonlinear output feedback design has been slower. First, nonlinear observers are available only for very restrictive classes of systems. Second, even when a nonlinear observer is available, it may not be applicable for output feedback design because the *separation principle* does not hold.

For systems in which the nonlinearities appear as functions of the measured output, the nonlinearity is canceled by an 'output injection' term. This class of systems has been characterized by Krener and Isidori (1983), Bestle and Zeitz (1983), Besançon (1999), among others. *Output injection observers* have been incorporated in observer-based control designs by Kanellakopoulos *et al.* (1992), Praly and Jiang (1993), Marino and Tomei (1993*a*), and, for stochastic nonlinear systems, by Deng and Krstić (1999).

A class of nonlinear observers by Thau (1973), Kou *et al.* (1975), Banks (1981), Tsinias (1989*a*), Yaz (1993), (Boyd *et al.*, 1994, Section 7.6), Raghavan and Hedrick (1994), and Rajamani (1998) require that the state-dependent non-linearities be globally Lipschitz, so that quadratic Lyapunov functions can be used for observer design.

A broader class of systems is characterized by linear dependence on unmeasured states. For this class, dynamic output feedback designs have been proposed by Praly (1992), Pomet *et al.* (1993), Marino and Tomei (1995), and Freeman and Kokotović (1996*c*).

For feedback linearizable systems Esfandiari and Khalil (1992), Khalil and Esfandiari (1993), Atassi and Khalil (1999) developed an output feedback design which achieves semiglobal stabilization and approximately recovers the performance of the underlying full state feedback. The key idea is to use a high-gain observer, but to pass the state estimates through saturation elements, thus avoiding the destabilizing effects of observer transients with large magnitudes. The high-gain observer has been employed in semiglobal output feedback designs by Teel and Praly (1995), Lin and Saberi (1995), Praly and Jiang (1998), and Isidori *et al.* (1999). Janković (1996) and Khalil (1996*a*) used the same approach in adaptive control.

Khalil's high-gain observer with saturation, along with the notion of *complete uniform observability* of Gauthier and Bornard (1981), led to the conceptually appealing 'separation theorem' by Teel and Praly (1994): If the equilibrium $x^\star$ is globally stabilizable by state feedback and the system is completely uniformly observable, then $x^\star$ is semiglobally stabilizable by dynamic output feedback. Extensions and interpretations of this result have been presented by Atassi and Khalil (1999), and (Isidori, 1999, Section 12.3).

To achieve global convergence of high-gain observers, Gauthier *et al.* (1992) resorted to a global Lipschitz condition - a common restriction in most global designs. In the absence of such a restriction, global stabilization by output feedback may not be possible, as shown by the counterexamples of Mazenc *et al.* (1994).

Arcak and Kokotović (1999) designed observers for systems with monotonic nonlinearities such as x^3, $exp(x)$, *etc.* Their approach is to represent the ob-

server error system as the feedback interconnection of a linear system and a time-varying sector nonlinearity. The convergence of the observer error to zero is then achieved by rendering the linear system SPR with the help of LMI computations as in the preceding section.

Isidori and Byrnes (1990) developed a nonlinear counterpart of the linear servomechanism design of Davison, Francis and Wonham, which incorporates an *internal model* of the disturbance. The internal model makes it possible to create and locally stabilize an *invariant manifold* on which the tracking error is zero. The local property restricts the disturbances and the initial conditions to be small. Huang and Rugh (1992) allowed large disturbances by restricting the exosystem to be slow. Khalil (1994), Mahmoud and Khalil (1996) and Khalil (1998) used a high-gain observer to solve the nonlinear servomechanism problem with arbitrarily large initial conditions. Developments in this area are treated in the book by Byrnes *et al.* (1997), and the survey by Byrnes and Isidori (1998).

4.7 Discrete-Time Problems

Much of nonlinear control research has been focused on continuous time models with continuous control signals. On the other hand, most implemented controllers are digital, that is, in discrete-time (sampling) and with finite word length (quantization). Discrete-time nonlinear control systems have been investigated by Sontag (1979), Monaco and Normand-Cyrot (1986, 1997), Grizzle (1985, 1993), Jakubczyk (1987), Jakubczyk and Sontag (1990), Nijmeijer and van der Schaft (1990), and many others. In discrete-time, geometric concepts lose their transparency and effectiveness. Considerable effort was made in the development of discrete-time observers, as in Moraal and Grizzle (1995).

Closer to the topics of this talk is the nonlinear passivity approach by Byrnes and Lin (1994), Lin and Byrnes (1995), which extends the linear results by Hitz and Anderson (1969). For the system

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{98}$$
$$y(k) = h(x(k)) + j(x(k))u(k) \, ,$$

which cannot be passive if $j(x(k)) = 0$, the passivation and stabilization results retain similarity with the continuous case, albeit in a more complicated form.

Discrete-time forwarding was developed by Yang *et al.* (1997), and Mazenc and Nijmeijer (1998). Constructive results for systems with polynomial nonlinearities were obtained by Nešić and Mareels (1998).

Further studies are likely to provide us with a wider range of nonlinear discrete-time design methods. However, a nagging question is when a model like (98) will be useful for sampled-data nonlinear control design. When (98) is an exact discrete-time model for a continuous plant, which is feasible for linear systems but few others, stabilization for (98) guarantees sampled-data stabilization. However, even for some linear systems, there exist controllers (parametrized with the sampling period T) which stabilize the Euler approximation for all $T > 0$ but destabilize the exact discrete-time model. Nešić *et al.* (1999) derived sufficient conditions for a controller, which stabilizes an approximate discrete-time model, to also stabilize the exact discrete-time model. However, a constructive design procedure for nonlinear sampled-data controllers with prescribed sampling period is yet to be developed.

This and the fact that sampling usually destroys many helpful structural properties motivate designs to remain in the continuous-time. Teel *et al.* (1998) showed that continuous-time ISS controllers, when implemented with sufficiently fast sampling, still achieve the same ISS property.

4.8   Other Topics

Among other important research areas, the three closest to the topics of this survey are briefly mentioned.

Model predictive control (MPC) is a collection of 'receding horizon' optimization methods in which the current control action is obtained by solving on-line, often approximately, an open-loop optimal control problem. The underlying theory of MPC methods and a growing body of results have been recently surveyed Mayne *et al.* (2000).

Nonholonomic systems, with applications to wheeled vehicles, mobile robots and space systems, are surveyed by Kolmanovsky and McClamroch (1995), Murray (1995), and Leonard (1998).

Magnitude and rate limits have been treated by optimization-based methods in Gilbert and Tan (1991), Megretski (1996), Shewchun and Feron (1997), and by *anti-windup* techniques in Teel and Kapoor (1997), and Teel (1998). A bibliography of some 150 papers is given in Bernstein and Michel (1995).

5   SELECTED APPLICATIONS

The much debated 'theory-applications gap' is a misleading term that overlooks the complex interplay between physics, invention and implementation,

on the one side, and theoretical abstractions, models and analytical designs, on the other side. A control invention is often ahead of its theoretical explanation, but, by abstracting the invention's common core, a theoretical analysis broadens its impact. Conversely, an analytical procedure, confronted with a new physical situation, often leads to an invention, which, in turn, is likely to expand or modify the procedure.

Such mutually enriching theory-applications transitions have been common in recent developments of nonlinear control, as illustrated by four representative examples.

5.1 Axial Compressors: L_gV Design

Experiments with a Rolls-Royce Viper turbojet reported by Freeman *et al.* (1998*a*), and similar studies by other authors, show that 'active control' may increase the stable operating range of axial flow compressors significantly. The early results of Liaw and Abed (1996) and Badmus *et al.* (1996) motivated Krstić *et al.* (1998) and Banaszuk and Krener (1997) to develop backstepping designs for throttle and bleed valve actuation, while Behnken and Murray (1997) and Protz and Paduano (1997) also investigated air injection.

A current study by Fontaine *et al.* (1999) for a compressor with a ring of individually actuated bleed valves, employs the following model:

$$\dot{\Phi} = \frac{1}{l_c}\left[\psi_c(\Phi) + \frac{1}{4}\psi_c''(\Phi)(a^2 + b^2) - \Psi + \Phi\bar{\Phi}_b + \frac{1}{2}(1 - \alpha\psi_c''(\Phi))(aC_a + bC_b)\right.$$
$$\left. -\alpha\left(\psi_c'(\Phi) + \frac{1}{4}\psi_c'''(\Phi)(a^2 + b^2)\right)\bar{\Phi}_b\right] \tag{99}$$

$$\dot{\Psi} = \frac{1}{4l_cB^2}\left[\Phi - \bar{\Phi}_b - \sqrt{\frac{2\Psi}{K_T}}\right] \tag{100}$$

$$\dot{a} = \frac{1}{\mu + m}\left[T_f a + T_g C_a - \lambda(b - \alpha C_b)\right] \tag{101}$$

$$\dot{b} = \frac{1}{\mu + m}\left[T_f b + T_g C_b + \lambda(a - \alpha C_a)\right] \tag{102}$$

where $\psi_c(\Phi) = k_0 + k_1\Phi + k_2\Phi^2 + k_3\Phi^3$ is the compressor characteristic and

$$T_g = \Phi - \alpha(\psi_c'(\Phi) + \frac{1}{8}\psi_c'''(\Phi)(a^2 + b^2))$$
$$T_f = (\psi_c'(\Phi) + \frac{1}{8}\psi_c'''(\Phi)(a^2 + b^2) + (1 - \alpha\psi_c''(\Phi))\bar{\Phi}_b - \frac{\alpha}{4}\psi_c'''(\Phi)(aC_a + bC_b)).$$

This model, derived by Liao (1997), is an approximation of the PDE model by

36

Moore and Greitzer (1986). Its four states are mass flow Φ, pressure rise Ψ, and the Fourier coefficients a and b of the first rotating stall mode. The controls are the first three terms of the Fourier series for the bleed flow: its mean $\bar{\Phi}_b$ and the coefficients C_a, C_b. These controls are to stabilize the equilibrium at the peak of $\psi_c(\Phi)$, that is, at the maximum achievable pressure rise.
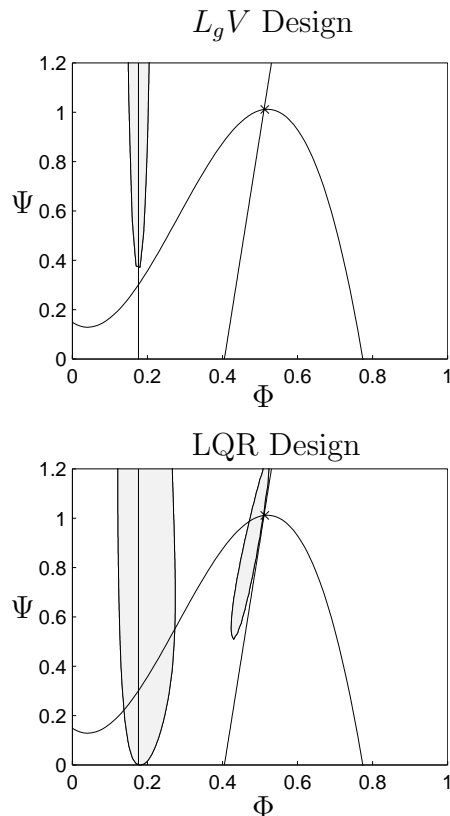

Fig. 6. Axial compressor, $\dot{V} > 0$ shaded.

For this model, Fontaine *et al.* (1999) demonstrated how a simple inverse optimal $L_g V$ design may dramatically enlarge the stability region achieved with a preliminary linear (optimal LQR) design. The quadratic optimal value function $V(x)$ of the LQR problem was used as the CLF for the $L_g V$ design. The stability properties of the two designs are judged by the regions where $\dot{V} > 0$. These regions, projected on the (Φ, Ψ)-plane with $(a^2 + b^2) = 0.01$, are shown as shaded areas in Figure 6. For the LQR design, the shaded strip with $\dot{V} > 0$ is unacceptably close to the equilibrium. This unstable region is due to a nonlinearity changing sign in the control input matrix. The $L_g V$ controller accommodates this destabilizing change of sign and, as shown in Figure 6, removes the region where $\dot{V} > 0$ from the area surrounding the equilibrium, thus providing a desired region of stability. Because of their extreme simplicity and potential effectiveness, $L_g V$ designs should be the first nonlinear designs to be tried.

37

Stringent emission and performance requirements have motivated the automotive industry to introduce additional actuators like exhaust gas recirculation (EGR) and variable geometry turbines (VGT) shown in Figure 7. Recirculating exhaust gases via the EGR valve into the intake manifold re-
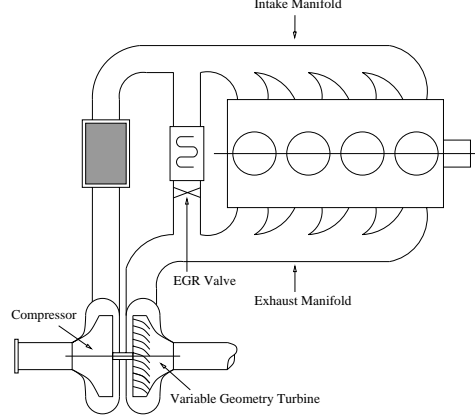

Fig. 7. Turbocharged Diesel Engine.

duces emissions. The exhaust gas flow through the VGT drives a turbocharger to improve engine performance. For this highly interactive system a seven-state nonlinear model was developed and validated at Ford by Kolmanovsky *et al.* (1997) and van Nieuwstadt *et al.* (1998). A simplified three-state model

$$
\begin{aligned}
\dot{p}_1 &= k_1(W_c - k_e p_1 + u) \\
\dot{p}_2 &= k_2(k_e p_1 + W_f - u - v) \\
\dot{P}_c &= \frac{1}{\tau}(-P_c + \eta_m P_t)\,,
\end{aligned}
\tag{103}
$$

was used by Janković *et al.* (1998) for a feedback passivation design. The states are the intake and exhaust manifold pressures p_1, p_2 and the compressor power, P_c, the two controls are $u = $ EGR flow, and $v = $ VGT flow. The significant nonlinearities in the compressor flow W_c and turbine power P_t are

$$
W_c = \frac{\eta_c}{T_a c_p} \frac{P_c}{p_1^\mu - 1}
\tag{104}
$$

$$
P_t = \eta_t c_p T_2 (1 - p_2^{-\mu}) v\,.
\tag{105}
$$

Regulation of the system outputs $y_1 = W_c - W_c^d$ and $y_2 = u - u_d$ to zero was made difficult by the instability of the zero dynamics. Instead, Janković used the statically equivalent outputs $\overline{y}_1 = y_1$ and $\overline{y}_2 = p_2 - p_{2e}$ with stable zero dynamics.

After a feedback transformation $[u, \ v]^T = T(w_1, w_2)$, with w_1 and w_2 as the new inputs, and with $\overline{y}_1$, $\overline{y}_2$, and $\overline{z} = p_1^\mu - p_{1e}^\mu$ as the new states, (103) becomes

$$\dot{\overline{z}} = q[-W_c^d \overline{z} + d_1 \overline{y}_1 + d_2 - (p_1^\mu - 1)(\tau w_1 + \frac{\tau b}{k_2} w_2)]$$

$$\dot{\overline{y}}_1 = -\frac{1}{\tau} \overline{y}_1 + w_1 \tag{106}$$

$$\dot{\overline{y}}_2 = w_2 ,$$

where $q = \frac{\mu p_1^{1-\mu} k_1}{\tau(a+b)(p_1^\mu - 1)}$, $d_1 = (p^\mu - 1)\tau b$, and $d_2 = \eta^* \frac{T_2}{T_a}(W_c^{-d} + W_f)(p_{2e}^{-\mu} - p_2^{-\mu})$. To arrive at this partially linear model, cancelations were made with $T(w_1, w_2)$. However, they were not implemented because the only purpose of the model (106) was to make a choice of a CLF simple. By inspection of (106) a convenient CLF is $V = c_1 \overline{y}_1^2 + c_2 \overline{y}_2^2 + c_3 \overline{z}^2$. This CLF is then expressed as a function of the states in the model (103). The dependence on the original controls u and v is thus recovered and a non-canceling passivation control law is designed as

$$u = -k L_{g_1} V + u_d \tag{107}$$
$$v = -k L_{g_2} V + W_c^d + W_f. \tag{108}$$

This control law achieves optimality for cost (6) with $R = I$, required to guarantee stability margins. After this control law was validated on the full order model, it was tested in a series of diesel engine experiments. They showed major improvements in both emissions and performance.

For the three-state model (103) an *indirect passivation* design by Larsen and Kokotović (1998) led to a comparable performance in simulations with the full order model. The main input-output pair was $(u, -\overline{y}_2)$ and the VGT flow v was used to stabilize the zero dynamics for that pair. This approach was both easy to understand and had practical appeal, because the use of VGT to stabilize the zero dynamics was physically meaningful.

5.3 Ship Control: Backstepping with Optimality

Advanced control designs for free-floating and moored ships are being developed, experimentally tested and implemented by Fossen (1994), Fossen and Grøvlen (1998), and Fossen and Strand (1999). A typical ship model for these designs is

$$\dot{\eta} = J(\eta)\nu$$
$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu = Bu \,, \tag{109}$$

where ν is the velocity vector decomposed in the body-fixed reference frame, η is the position/attitude vector decomposed in the Earth-fixed coordinates and u is a vector of control inputs: azimuth thrusters, main propellers, and tunnel thrusters. In *vectorial backstepping* Fossen and Berge (1997) used $J(\nu)\eta$ as the first virtual control, and the state feedback design is completed at the second step with the actual control vector. For output feedback designs observer backstepping was developed by Fossen and Grøvlen (1998), and Fossen and Strand (1999). Recently Strand *et al.* (1998) combined optimal linear controllers, that perform well locally, with backstepping controllers that have inverse optimal properties in large operating regions. The resulting design was experimentally tested in the NUST Laboratory on a model ship with encouraging results.

5.4 Induction Motor: Adaptive Control

Electric machines, especially synchronous generators and induction motors, have long been objects of nonlinear control. *Field orientation control* of Blaschke (1972), Leonhard (1996), is a prime example of an invention ahead of theory. In the late 1980's and in the 1990's, various designs of electrical and electromechanical systems employed state or observer-based feedback linearization, backstepping, passivation and adaptive control. About twenty such designs are described, along with experiments, in the book by Dawson *et al.* (1998), with a rich bibliography. Diverse passivation designs can be found in the book by Ortega *et al.* (1998) along with many references. Sensorless motor control, which is of major commercial interest, is a topic of many papers including Shouse and Taylor (1998), and Chang and Fu (1998).

A good induction motor example is the adaptive output feedback design of Marino *et al.* (1996, 1999). In its simpler 1996 version, the usual 5-state voltage-controlled model is first reduced, via singular perturbations, to the 3-state current-controlled model

$$\frac{d\omega}{dt} = \mu(\psi_a i_b - \psi_b i_a) - \frac{T_L}{J}$$
$$\frac{d\psi_a}{dt} = -\frac{R_r}{L_r}\psi_a - \omega\psi_b + R_r\frac{M}{L_r}i_a \tag{110}$$
$$\frac{d\psi_b}{dt} = \omega\psi_a - \frac{R_r}{L_r}\psi_b + R_r\frac{M}{L_r}i_b \,,$$

where ω is the rotor speed, (ψ_a, ψ_b) are the rotor fluxes and (i_a, i_b) are the stator currents.

Their initial design was with state feedback using a CLF V quadratic in the tracking errors $\tilde{\omega} = \omega - \omega_r$, $\tilde{\psi} = \psi_a^2 + \psi_b^2 - \psi_r^2$. With the control law chosen to render $\dot{V}$ negative, exponential convergence $\tilde{\omega}(t) \to 0$, $\tilde{\psi}(t) \to 0$ was achieved. At the next design step, convergent flux estimates $\hat{\psi}_a$, $\hat{\psi}_b$ were obtained from an observer mimicking the last two equations of (110). Then the CLF was augmented with the squares of the flux estimation errors, and used to design a control law employing $\hat{\psi}_a$, $\hat{\psi}_b$ instead of ψ_a, ψ_b. An adaptive update law was added for the constant but unknown load torque T_L. At the final and most complex step, an identifier was designed for the rotor resistance R_r, slowly varying due to temperature changes. A good estimate of R_r was needed to ensure good flux estimates $\hat{\psi}_a$ and $\hat{\psi}_b$. Experimental results were reported and interpreted, indicating that the design achieved its stated objectives.

6 Looking Ahead

Constructive trends in nonlinear control were barely discernible in a survey completed fifteen years ago by Kokotović (1985). One prediction, which was then easy to make, was that nonlinear geometric concepts were soon to become engineering tools. What was harder to predict, but fortunately occurred in a span of ten years, was the activation of stability, optimality and passivity concepts, and even dynamic games, all of which joined the geometric methods to form constructive nonlinear designs described in this survey.

The constructive trend will doubtless continue, with further fusion of its ingredients into structure-specific procedures applicable to broader classes of systems. This process has already started for structures induced by physical laws for electromechanical systems, with new challenges at micro and nano-scales.

Constructive procedures have been developed for only a few output feedback problems. This is an area where discoveries of new structures may lead to significant breakthroughs.

Physically motivated characterizations of nonlinear uncertainties, that is, unmodeled dynamics, deterministic and stochastic disturbances, are needed to help robustify the constructive procedures, without undue conservativeness. To reduce complexity of feedback designs, attention must be paid to structuring and simplification of models.

Most of the surveyed tools and design procedures are analytical, while only a few relied on LMI computations. Symbolic and numerical procedures will strengthen analytical design methods.

41

Extensions of constructive procedures described in this survey to PDE models of infinite dimensional systems promise to solve open problems of theoretical and practical interest. First steps in this direction include a rotating body beam stabilizer by Coron and d'Andréa Novel (1998), and flow control designs by Liu and Krstić (2000a,b).

Nonlinear control designs are increasingly important in a wide range of technologies. With a solid knowledge of nonlinear control, new generations of engineers will be better equipped for new creative tasks.

Acknowledgements

References

Aizerman, M.A. and F.R. Gantmacher (1964). *Absolute stability of Regulator Systems*. Holden-Day. San Francisco. Translated from the Russian original, Akad. Nauk SSSR, Moscow, 1963.

Anderson, B.D.O. (1967). A system theory criterion for positive real matrices. *SIAM Journal of Control and Optimization* **5**, 171–182.

Anderson, B.D.O. and J.B. Moore (1971). *Optimal Control, Linear Quadratic Methods*. Prentice Hall. Englewood Cliffs, NJ. Second edition: 1990.

Anderson, B.D.O. and S. Vongpanitlerd (1973). *Network Analysis and Synthesis*. Prentice Hall. Englewood Cliffs, NJ.

Angeli, D., E.D. Sontag and Y. Wang (1998). A remark on integral input to state stability. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 2491–2496.

Annaswamy, A.M., A.P. Loh and F.P. Skantze (1998). Adaptive control of continuous time systems with convex/concave parametrization. *Automatica* **34**, 33–49.

Arcak, M., A. Teel and P. Kokotović (2000). Robust nested saturation redesign for systems with input unmodeled dynamics. In: *Proceedings of the 2000 American Control Conference*. Chicago, IL. pp. 150–154.

Arcak, M. and P.V. Kokotović (1999). Nonlinear observers: A circle criterion design. In: *Proceedings of the 38th IEEE Conference on Decision and Control*. Phoenix, AZ. pp. 4872–4876.

Artstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Analysis* **7**, 1163–1173.

Atassi, A.N. and H.K. Khalil (1999). A separation principle for the stabilization of a class of nonlinear systems. *IEEE Transactions on Automatic Control* **44**, 1672–1687.

Athans, M. and P.L. Falb (1965). *Optimal Control: An Introduction to the Theory and its Applications*. McGraw-Hill.

Badmus, O., S. Chowdhury and C. Nett (1996). Nonlinear control of surge in axial compression. *Automatica* **32**, 59–70.

Ball, J.A. and A.J. van der Schaft (1996). J-inner-outer factorization, J-spectral factorization and robust control for nonlinear systems. *IEEE Transactions on Automatic Control* **41**, 379–392.

Ball, J.A. and J.W. Helton (1992). $\mathcal{H}_\infty$ control for stable nonlinear plants. *Mathematics of Control, Signals, and Systems* **5**, 233–262.

Ball, J.A., J.W. Helton and M. Walker (1993). $\mathcal{H}_\infty$ control for nonlinear systems via output feedback. *IEEE Transactions on Automatic Control* **38**, 546–559.

Banaszuk, A. and A. Krener (1997). Design of controllers for MG3 compressor models with general characteristics using graph backstepping. In: *Proceedings of the 1997 American Control Conference*. Albuquerque, NM. pp. 977–981.

Banks, S.P. (1981). A note on non-linear observers. *International Journal of Control* **34**, 185–190.

Barbashin, E.A. (1967). *Introduction to the Theory of Stability*. Nauka. Moscow. (in Russian), English translation: Wolters-Noordhoff Publishing, 1970.

Başar, T. and G.J. Olsder (1982). *Dynamic Noncooperative Game Theory*. Academic Press.

Başar, T. and M. Mintz (1972). Minimax terminal state estimation for linear plants with unknown forcing functions. *International Journal of Control* **16**, 49–70.

Başar, T. and P. Bernhard (1995). $\mathcal{H}_\infty$ *Optimal Control and Related Minimax Design Problems*. second ed.. Birkhauser. Boston.

Behnken, B. and R. Murray (1997). Combined air injection control of rotating stall and bleed valve control of surge. In: *Proceedings of the 1997 American Control Conference*. Albuquerque, NM. pp. 987–992.

Bernstein, D.S. and A.N. Michel (1995). A chronological bibliography on saturating actuators. *International Journal of Robust and Nonlinear Control* **5**, 375–381.

Bertsekas, D.P. and I.B. Rhodes (1971). Recursive state estimation for a set membership description of uncertainty. *IEEE Transactions on Automatic Control* **16**, 117–128.

Bertsekas, D.P. and I.B. Rhodes (1973). Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems. *IEEE Transactions on Automatic Control* **18**, 117–123.

Besançon, G. (1999). On output transformations for state linearization up to output injection. *IEEE Transactions on Automatic Control* **44**, 1975–1981.

Bestle, D. and M. Zeitz (1983). Canonical form observer design for non-linear time-variable systems. *International Journal of Control* **38**, 419–431.

Blaschke, F. (1972). The principle of field orientation applied to the new transvector closed-loop control system for rotating field machines. *Siemens-Review* **39**, 217–220.

Bošković, J.D. (1998). Adaptive control of a class of nonlinearly parametrized plants. *IEEE Transactions on Automatic Control* **43**, 930–934.

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. Vol. 15 of *SIAM Studies in Applied Mathematics*. SIAM. Philadelphia, PA.

Braslavsky, J.H. and R.H. Middleton (1996). Global and semiglobal stabilizability in certain cascade nonlinear systems. *IEEE Transactions on Automatic Control* **41**, 876–880.

Brockett, R.W. (1964). On the stability of nonlinear feedback systems. *IEEE Transactions on Applications and Industry* **83**, 443–448.

Brockett, R.W. (1966). The status of stability theory for deterministic systems. *IEEE Transactions on Automatic Control* **11**, 596–606.

Brockett, R.W. and J.L.Willems (1965). Frequency domain stability criteria-Parts I and II. *IEEE Transactions on Automatic Control* **10**, 255–261, 407–413.

Bryson, A.E. and Y.-C. Ho (1969). *Applied Optimal Control*. Blaisdel Publishing Company.

Byrnes, C.I., A. Isidori and J.C. Willems (1991). Passivity, feedback equivalence, and global stabilization of minimum phase systems. *IEEE Transactions on Automatic Control* **36**, 1228–1240.

Byrnes, C.I. and A. Isidori (1989). New results and examples in nonlinear feedback stabilization. *Systems and Control Letters* **12**, 437–442.

Byrnes, C.I. and A. Isidori (1991). Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control* **36**, 1122–1137.

Byrnes, C.I. and A. Isidori (1998). Output regulation for nonlinear systems: an overview. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 3069–3074.

Byrnes, C.I. and W. Lin (1994). Losslessness, feedback equivalence and the global stabilization of discrete-time nonlinear systems. *IEEE Transactions on Automatic Control* **39**, 83–97.

Byrnes, C.I., F. Delli Priscoli and A. Isidori (1997). *Output Regulation of Uncertain Nonlinear Systems*. Birkhauser. Boston.

Chang, R.-J. and L.-C. Fu (1998). Nonlinear adaptive sensorless speed control of induction motors. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 965–971.

Chetaev, N.G. (1955). *Stability of Motion*. GITTL. Moscow.

Cho, Y.-S. and K.S. Narendra (1968). An off-axis circle criterion for the stability of feedback systems with a monotonic nonlinearity. *IEEE Transactions*

on Automatic Control **13**, 413–416.

Corless, M.J. and G. Leitmann (1981). Continuous state feedback guaranteeing uniform ultimate boundedness. *IEEE Transactions on Automatic Control* **26**, 1139–1144.

Coron, J.-M. and B. d'Andréa Novel (1998). Stabilization of a rotating body beam without damping. *IEEE Transactions on Automatic Control* **43**, 608–618.

Crandall, M.G., L.C. Evans and P.L. Lions (1984). Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society* **282**, 487–502.

Dawson, D.M., J. Hu and T.C. Burg (1998). *Nonlinear Control of Electric Machinery*. Marcel Dekker Inc.

Deng, H. and M. Krstić (1999). Output-feedback stochastic nonlinear stabilization. *IEEE Transactions on Automatic Control* **44**, 328–333.

Desoer, C.A. and M. Vidyasagar (1975). *Feedback Systems: Input-Output Properties*. Academic Press. New York.

Didinsky, G. and T. Başar (1992). Design of minimax controllers for linear systems with non-zero initial states under specified information structures. *International Journal of Robust and Nonlinear Control* **2**, 1–30.

Didinsky, G., T. Başar and P. Bernhard (1993). Structural properties of minimax controllers for a class of differential games arising in nonlinear $\mathcal{H}_\infty$ control.. *Systems and Control Letters* **21**, 433–441.

Didinsky, G., Z. Pan and T. Başar (1995). Parameter identification for uncertain plants using $\mathcal{H}_\infty$ methods. *Automatica* **31**, 1227–1250.

Dorato, P. and R.F. Drenick (1966). Optimality, insensitivity and game theory. In: *Sensitivity Methods in Control Theory*. pp. 78–102. Pergamon Press. New York, NY.

Doyle, J.C., K. Glover, P. Khargonekar and B.A. Francis (1989). State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *IEEE Transactions on Automatic Control* **34**, 831–847.

Emelyanov, S.V. (1967). *Variable Structure Control Systems*. Nauka. Moscow.

Esfandiari, F. and H.K. Khalil (1992). Output feedback stabilization of fully linearizable systems. *International Journal of Control* **56**, 1007–1037.

Ezal, K., Z. Pan and P.V. Kokotović (2000). Locally optimal and robust backstepping design. *IEEE Transactions on Automatic Control* **45**, 260–271.

Feuer, A. and A.S. Morse (1978). Adaptive control of single-input single-output linear systems. *IEEE Transactions on Automatic Control* **23**, 557–569.

Filippov, A.F. (1964). Differential equations with discontinuous right-hand side. *American Mathematical Society translations* **42**, 199–231.

Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers. Netherlands.

Fontaine, D., S. Liao, P. Kokotović and J. Paduano (1999). Two dimensional, nonlinear control of an axial flow compressor. In: *Proceedings of the IEEE International Conference on Control Applications*. Hawaii. pp. 921–926.

Fossen, T.I. (1994). *Guidance and Control of Ocean Vehicles*. John Wiley & Sons, Inc.. Chicester, England.

Fossen, T.I. and Å. Grøvlen (1998). Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping. *IEEE Transactions on Control Systems Technology* **6**, 121–128.

Fossen, T.I. and J.P. Strand (1999). Passive nonlinear observer design for ships using Lyapunov methods: full-scale experiments with a supply vessel. *Automatica* **35**, 3–16.

Fossen, T.I. and S.P. Berge (1997). Nonlinear vectorial backstepping design for global exponential tracking of marine vessels in the presence of actuator dynamics. In: *Proceedings of the 36th IEEE Conference on Decision and Control*. San Diego, CA. pp. 4237–4242.

Fradkov, A. and D. Hill (1998). Exponential feedback passivity and stabilizability of nonlinear systems. *Automatica* **34**, 697–703.

Fradkov, A.L. (1976). Quadratic Lyapunov functions in the adaptive stability problem of a linear dynamic target. *Siberian Math. Journal* pp. 341–348.

Freeman, C., A.G. Wilson, I.J. Day and M.A. Swinbanks (1998a). Experiments in active control of stall on an aeroengine gas turbine. *Transactions of the ASME, Journal of Turbomachinery* **120**, 637–647.

Freeman, R.A. and L. Praly (1998). Integrator backstepping for bounded controls and control rates. *IEEE Transactions on Automatic Control* **43**, 258–262.

Freeman, R.A. and P.V. Kokotović (1992). Backstepping design of robust controllers for a class of nonlinear systems. In: *Preprints of 2nd IFAC Nonlinear Control Systems Design Symposium*. Bordeaux, France. pp. 307–312.

Freeman, R.A. and P.V. Kokotović (1993). Design of softer robust nonlinear control laws. *Automatica* **29**, 1425–1437.

Freeman, R.A. and P.V. Kokotović (1996a). Inverse optimality in robust stabilization. *SIAM Journal of Control and Optimization* **34**, 1365–1391.

Freeman, R.A. and P.V. Kokotović (1996b). *Robust Nonlinear Control Design, State-Space and Lyapunov Techniques*. Birkhauser. Boston.

Freeman, R.A. and P.V. Kokotović (1996c). Tracking controllers for systems linear in the unmeasured states. *Automatica* **32**, 735–746.

Freeman, R.A., M. Krstić and P.V. Kokotović (1998b). Robustness of adaptive nonlinear control to bounded uncertainties. *Automatica* **34**, 1227–1230.

Gauthier, J.-P. and G. Bornard (1981). Observability for any $u(t)$ of a class of nonlinear systems. *IEEE Transactions on Automatic Control* **26**, 922–926.

Gauthier, J.P., H. Hammouri and S. Othman (1992). A simple observer for nonlinear systems, applications to bioreactors. *IEEE Transactions on Automatic Control* **37**, 875–880.

Gilbert, E.G. and K.T. Tan (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control* **36**, 1008–1020.

Glad, S.T. (1984). On the gain margin of nonlinear and optimal regulators. *IEEE Transactions on Automatic Control* **29**, 615–620.

Grizzle, J.W. (1985). Controlled invariance for discrete-time nonlinear systems with an application to the decoupling problem. *IEEE Transactions on Automatic Control* **30**, 868–874.

Grizzle, J.W. (1993). A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM Journal of Control and Optimization* **31**, 1026–1044.

Grognard, F., R. Sepulchre and G. Bastin (1999). Global stabilization of feedforward systems with exponentially unstable Jacobian linearization. *Systems and Control Letters* **37**, 107–115.

Gutman, S. (1979). Uncertain dynamical systems-Lyapunov min-max approach. *IEEE Transactions on Automatic Control* **24**, 437–443.

Hahn, W. (1967). *Stability of Motion*. Springer-Verlag. Berlin.

Hamzi, B. and L. Praly (1999). Ignored input dynamics and a new characterization of control Lyapunov functions. In: *Proceedings of the 5th European Control Conference*. Karlsruhe, Germany.

Helton, J.W. and M.R. James (1999). *Extending $\mathcal{H}_\infty$ Control to Nonlinear Systems*. SIAM Frontiers in Applied Mathematics.

Hill, D. (1991). A generalisation of the small-gain theorem for nonlinear feedback systems. *Automatica* **27**, 1043–1045.

Hill, D. and P. Moylan (1976). The stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control* **21**(5), 708–711.

Hill, D. and P. Moylan (1977). Stability results for nonlinear feedback systems. *Automatica* **13**, 377–382.

Hill, D. and P. Moylan (1980*a*). Connections between finite gain and asymptotic stability. *IEEE Transactions on Automatic Control* **25**, 931–936.

Hill, D. and P. Moylan (1980*b*). Dissipative dynamical systems: Basic input-output and state properties. *Journal of Franklin Institute* **309**, 327–357.

Hill, D. and P. Moylan (1983). General instability results for interconnected systems. *SIAM Journal of Control and Optimization* **21**, 256–279.

Hitz, B.E. and B.D.O. Anderson (1969). Discrete positive-real functions and their application to system stability. *Proceedings of the Institution of Electrical Engineers* **116**, 153–155.

Huang, J. and W.J. Rugh (1992). Stabilization on zero-error manifold and the nonlinear servomechanism problem. *IEEE Transactions on Automatic Control* **37**, 1009–1013.

Ikhouane, F. and M. Krstić (1998). Robustness of the tuning functions adaptive backstepping design for linear systems. *IEEE Transactions on Automatic Control* **43**, 431–437.

Ioannou, P.A. and J. Sun (1996). *Robust Adaptive Control*. Prentice Hall. Englewood Cliffs, NJ.

Isaacs, R. (1975). *Differential Games*. Kruger Publishing Company. Huntington, NY. First Edition: Wiley, NY, 1965.

Isidori, A. (1995). *Nonlinear Control Systems*. third ed.. Springer-Verlag. Berlin.

Isidori, A. (1996*a*). Global almost disturbance decoupling with stability for

non-minimum-phase single-input single-output nonlinear systems. *Systems and Control Letters* **28**, 115–122.

Isidori, A. (1996*b*). A note on almost disturbance decoupling for nonlinear minimum phase systems. *Systems and Control Letters* **27**, 191–194.

Isidori, A. (1999). *Nonlinear Control Systems II*. Springer-Verlag. London.

Isidori, A., A. Teel and L. Praly (1999). Dynamic UCO controllers and semiglobal stabilization of uncertain nonminimum phase systems by output feedback. In: *New Directions in Nonlinear Observer Design* (H. Nijmeijer and T.I. Fossen, Eds.). pp. 335–350. Springer-Verlag.

Isidori, A. and A. Astolfi (1992). Disturbance attenuation and $\mathcal{H}_\infty$ control via measurement feedback in nonlinear systems. *IEEE Transactions on Automatic Control* **37**, 1283–1293.

Isidori, A. and C.I. Byrnes (1990). Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control* **35**, 131–140.

Isidori, A. and W. Kang (1995). $\mathcal{H}_\infty$ control via measurement feedback for general nonlinear systems. *IEEE Transactions on Automatic Control* **40**, 466–472.

Jacobson, D.H. (1977). *Extensions of Linear-Quadratic Control, optimization and matrix theory*. Academic Press. New York.

Jakubczyk, B. (1987). Feedback linearization of discrete-time systems. *Systems and Control Letters* **9**, 411–416.

Jakubczyk, B. and E.D. Sontag (1990). Controlability of nonlinear discrete-time systems: A Lie-algebraic approach. *SIAM Journal of Control and Optimization* **28**, 1–37.

James, M.R. and J.S. Baras (1995). Robust $\mathcal{H}_\infty$ output feedback control for nonlinear systems. *IEEE Transactions on Automatic Control* **40**, 1007–1017.

Janković, M. (1996). Adaptive output feedback control of nonlinear feedback linearizable systems. *International Journal of Adaptive Control and Signal Processing* **10**, 1–18.

Janković, M., M. Janković and I. Kolmanovsky (1998). Constructive Lyapunov control design for turbocharged diesel engines. In: *Proceedings of the 1998 American Control Conference*. Philadelphia, PA. pp. 1389–1394.

Janković, M., M. Larsen and P.V. Kokotović (1999*a*). Master-slave passivity design for stabilization of nonlinear systems. In: *Proceedings of the 18th American Control Conference*. San Diego, CA. pp. 769–773.

Janković, M., R. Sepulchre and P. Kokotović (1999*b*). CLF based designs with robustness to dynamic input uncertainties. *Systems and Control Letters* **37**, 45–54.

Janković, M., R. Sepulchre and P.V. Kokotović (1996). Constructive Lyapunov stabilization of nonlinear cascade systems. *IEEE Transactions on Automatic Control* **41**, 1723–1736.

Jiang, Z.-P. and H. Nijmeijer (1997). Tracking control of mobile robots: a case study in backstepping. *Automatica* **33**, 1393–1399.

Jiang, Z.-P. and I. Mareels (1997). A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on*

Automatic Control **42**, 292–308.

Jiang, Z.-P. and L. Praly (1991). Iterative designs of adaptive controllers for systems with nonlinear integrators. In: *Proceedings of the 30th IEEE Conference on Decision and Control*. Brighton, UK. pp. 2482–2487.

Jiang, Z.-P. and L. Praly (1998). Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. *Automatica* **34**, 835–840.

Jiang, Z.-P., A.R. Teel and L. Praly (1994). Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems* **7**, 95–120.

Jurdjević, V. and J.P. Quinn (1978). Controllability and stability. *Journal of Differential Equations* **28**, 381–389.

Jury, E.I. and B.W. Lee (1964). On the stability of a class of nonlinear sampled-data systems. *IEEE Transactions on Automatic Control* **9**, 51–61.

Kalman, R. (1963). Lyapunov functions for the problem of Lur'e in automatic control. *Proceedings of the National Academy of Sciences of the United States of America* **49**, 201–205.

Kalman, R. (1964). When is a linear control system optimal?. *Transactions of the ASME, Series D, Journal of basic engineering* **86**, 1–10.

Kalman, R. and J. Bertram (1960). Control system analysis and design via the second method of Lyapunov, Part I, Continuous-Time Systems. *Transactions of the ASME, Series D, Journal of basic engineering* **82**, 371–393.

Kalman, R.E. and G. Szegö (1963). Sur la stabilité d'un système d'équation aux différences finies. *CR Acad. Sci.* Paris **257**, 388–390.

Kanellakopoulos, I., P.V. Kokotović and A.S. Morse (1991*a*). Adaptive feedback linearization of nonlinear systems. In: *Foundations of Adaptive Control* (P.V. Kokotović, Ed.). pp. 311–346. Springer-Verlag. Berlin.

Kanellakopoulos, I., P.V. Kokotović and A.S. Morse (1991*b*). Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control* **36**, 1241–1253.

Kanellakopoulos, I., P.V. Kokotović and A.S. Morse (1992). A toolkit for nonlinear feedback design. *Systems and Control Letters* **18**, 83–92.

Kanellakopoulos, I., P.V. Kokotović and R. Marino (1991*c*). An extended direct scheme for robust adaptive nonlinear control. *Automatica* **27**, 247–255.

Kapila, V. and W. Haddad (1996). A multivariable extension of the Tsypkin criterion using a Lyapunov-function approach. *IEEE Transactions on Automatic Control* **41**, 149–152.

Khalil, H.K. (1994). Robust servomechanism output feedback controllers for a class of feedback linearizable systems. *Automatica* **30**, 1587–1599.

Khalil, H.K. (1996*a*). Adaptive output feedback control of nonlinear systems represented by input-output models. *IEEE Transactions on Automatic Control* **41**, 177–188.

Khalil, H.K. (1996*b*). *Nonlinear Systems*. second ed.. Prentice Hall. Englewood Cliffs, NJ.

Khalil, H.K. (1998). On the design of robust servomechanisms for minimum phase nonlinear systems. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 3075–3080.

Khalil, H.K. and F. Esfandiari (1993). Semiglobal stabilization of a class of nonlinear systems using output feedback. *IEEE Transactions on Automatic Control* **38**, 1412–1415.

Kojić, A., A.M. Annaswamy, A.-P. Loh and R. Lozano (1998). Adaptive control of a class of second order nonlinear systems with convex/concave parametrization. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 2849–2855.

Kokotović, P.V. (1985). Recent trends in feedback design: an overview. *Automatica* **21**, 225–236.

Kokotović, P.V. (1992). The joy of feedback: Nonlinear and adaptive. *IEEE Control Systems Magazine* **12**, 7–17.

Kokotović, P.V. and H.J. Sussmann (1989). A positive real condition for global stabilization of nonlinear systems. *Systems and Control Letters* **19**, 177–185.

Kokotović, P.V. and I. Kanellakopoulos (1990). Adaptive nonlinear control: A critical appraisal. In: *Proceedings of the 6th Yale Workshop on Adaptive and Learning Systems*. New Haven, CT. pp. 1–6.

Kolmanovsky, I. and N.H. McClamroch (1995). Developments in nonholonomic control problems. *IEEE Control Systems Magazine* **15**, 20–36.

Kolmanovsky, I., P. Moraal, M. van Nieuwstadt and A. Stefanopoulou (1997). Issues in modeling and control of intake flow in variable geometry turbocharged engines. In: *Proceedings of the 18th IFIP Conference on System Modeling and Optimization*. Detroit, MI.

Kou, S.R., D.L. Elliott and T.J. Tarn (1975). Exponential observers for nonlinear dynamic systems. *Information and Control* **29**, 204–216.

Krasovskii, A.N. and N.N. Krasovskii (1995). *Control Under Lack of Information*. Birkhauser. Boston.

Krasovskii, N.N. (1959). *Some Problems of the Stability Theory*. Fizmatgiz.

Krasovskii, N.N. and A.I. Subbotin (1988). *Game-Theoretical Control Problems*. Springer-Verlag. New York.

Krasovsky, A.A. (1971). A new solution to the problem of a control system analytical design. *Automatica* **7**, 45–50.

Krener, A.J. (1994). Necessary and sufficient conditions for nonlinear worst case $\mathcal{H}_\infty$ control and estimation. *Journal of Mathematical Systems, Estimation, and Control* **4**, 485–488.

Krener, A.J. and A. Isidori (1983). Linearization by output injection and nonlinear observers. *Systems and Control Letters* **3**, 47–52.

Krstić, M. and H. Deng (1998). *Stabilization of Nonlinear Uncertain Systems*. Springer-Verlag. New York.

Krstić, M. and P. Kokotović (1995). Adaptive nonlinear design with controller-identifier separation and swapping. *IEEE Transactions on Automatic Control* **40**, 426–441.

Krstić, M. and P. Kokotović (1996). Modular approach to adaptive stabilization. *Automatica* **32**, 625–629.

Krstić, M. and Z. Li (1998). Inverse optimal design of input-to-state stabilizing nonlinear controllers. *IEEE Transactions on Automatic Control* **43**, 336–

351.

Krstić, M., D. Fontaine, P. Kokotović and J. Paduano (1998). Useful nonlinearities and global bifurcation control of jet engine stall and surge. *IEEE Transactions on Automatic Control* **43**, 1739–1745.

Krstić, M., I. Kanellakopoulos and P. Kokotović (1995). *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc.. New York.

Krstić, M., I. Kanellakopoulos and P.V. Kokotović (1992). Adaptive nonlinear control without overparametrization. *Systems and Control Letters* **43**, 336–351.

Krstić, M., I. Kanellakopoulos and P.V. Kokotović (1994). Nonlinear design of adaptive controllers for linear systems. *IEEE Transactions on Automatic Control* **39**, 738–752.

Krstić, M., J. Sun and P. Kokotović (1996). Robust control of nonlinear systems with input unmodeled dynamics. *IEEE Transactions on Automatic Control* **41**, 913–920.

Kurzweil, J. (1956). On the inversion of Liapunov's second theorem on stability of motion. *American Mathematical Society translations* **24**, 19–77.

Larsen, M. and P. Kokotović (1998). Passivation design for a turbocharged diesel engine model. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 1535–1541.

LaSalle, J. and S. Lefschetz (1961). *Stability by Liapunov's Direct Method with Applications*. Academic Press. New York.

LaSalle, J.P. (1968). Stability theory for ordinary differential equations. *Journal of Differential Equations* **4**, 57–65.

Lee, E.B. and L. Markus (1967). *Foundations of Optimal Control Theory*. John Wiley & Sons, Inc.. New York.

Lefschetz, S. (1965). *Stability of Nonlinear Control Systems*. Academic Press. New York.

Leonard, N.E. (1998). Mechanics and nonlinear control: Making underwater vehicles ride and glide. In: *Preprints of the 4th IFAC Nonlinear Control Systems Design Symposium*. Enschede, Netherlands. pp. 1–6.

Leonhard, W. (1996). *Control of Electrical Drives*. second ed.. Springer-Verlag. Berlin.

Liao, S. (1997). Modeling interstage bleed valves in axial flow compressors. Technical Report PRET M77-2-19. MIT Gas Turbine Laboratory.

Liaw, D.-C. and E. Abed (1996). Active control of compressor stall inception: A bifurcation-theoretic approach. *Automatica* **32**, 109–115.

Lin, J.S. and I. Kanellakopoulos (1998). Nonlinearities enhance parameter convergence in strict-feedback systems. *IEEE Transactions on Automatic Control* **43**, 204–223.

Lin, W. and C. Qian (1998). New results on global stabilization of feedforward systems via small feedback. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 873–87.

Lin, W. and C.I. Byrnes (1995). Passivity and absolute stabilization of a class of discrete-time nonlinear systems. *Automatica* **31**, 263–268.

Lin, Z. and A. Saberi (1995). Robust semi-global stabilization of minimum-phase input-output linearizable systems via partial state and output feedback. *IEEE Transactions on Automatic Control* **40**, 1029–1041.

Liu, W.-J. and M. Krstić (2000*a*). Coping with actuator dynamics using backstepping for boundary stabilization of Burger's equation. In: *Proceedings of the 2000 American Control Conference*. Chicago, IL. pp. 4262–4268.

Liu, W.-J. and M. Krstić (2000*b*). Estimation of viscosity and control strengthening in boundary stabilization of Burger's equation. In: *Proceedings of the 2000 American Control Conference*. Chicago, IL. pp. 2295–2299.

Lozano-Leal, R. and S.M. Joshi (1990). Strictly positive real transfer functions revisited. *IEEE Transactions on Automatic Control* **35**, 1243–1245.

Lurie, A.I. (1951). *Some Nonlinear Problems in the Theory of Automatic Control*. Gostekhizdat. Moscow.

Mageirou, E.F. (1976). Values and strategies for infinite durations linear quadratic games. *IEEE Transactions on Automatic Control* **21**, 547–550.

Mahmoud, N.A. and H.K. Khalil (1996). Asymptotic regulation of minimum phase nonlinear systems using output feedback. *IEEE Transactions on Automatic Control* **41**, 1402–1413.

Malkin, I.G. (1952). *The Theory of Stability of Motion*. Gostekhizdat. Moscow.

Mareels, I.M.Y. and D.J. Hill (1992). Monotone stability of nonlinear feedback systems. *Journal of Mathematical Systems, Estimation, and Control* **2**, 275–291.

Marino, R. and P. Tomei (1993*a*). Global adaptive output-feedback control of nonlinear systems. Parts I-II. *IEEE Transactions on Automatic Control* **38**, 17–32, 33–49.

Marino, R. and P. Tomei (1993*b*). Robust stabilization of feedback linearizable time-varying uncertain systems. *Automatica* **29**, 181–189.

Marino, R. and P. Tomei (1995). *Nonlinear Control Design: Geometric, Adaptive and Robust*. Prentice Hall. London.

Marino, R., S. Peresada and P. Tomei (1996). Output feedback control of current-fed induction motors with unknown rotor resistance. *IEEE Transactions on Control Systems Technology* **4**, 336–347.

Marino, R., S. Peresada and P. Tomei (1999). Global adaptive output feedback control of induction motors with uncertain rotor resistance. *IEEE Transactions on Automatic Control* **44**, 967–983.

Marino, R., W. Respondek, A.J. van der Schaft and P. Tomei (1994). Nonlinear $\mathcal{H}_\infty$ almost disturbance decoupling. *Systems and Control Letters* **23**, 159–168.

Massera, J.L. (1956). Contributions to stability theory. *Annals of Mathematics* **64**, 182–206.

Mayne, D.Q., J.B. Rawlings, C.V. Rao and P.O.M. Scokaert (2000). Constrained model predictive control: Stability and optimality. *Automatica* **36**, 789–814.

Mazenc, F. (1997). Stabilization of feedforward systems approximated by a nonlinear chain of integrators. *Systems and Control Letters* **32**, 223–229.

Mazenc, F. and H. Nijmeijer (1998). Forwarding in discrete-time nonlinear systems. *International Journal of Control* **71**, 823–837.

Mazenc, F. and L. Praly (1996). Adding integrations, saturated controls and stabilization for feedforward systems. *IEEE Transactions on Automatic Control* **41**, 1559–1578.

Mazenc, F., L. Praly and W.P. Dayawansa (1994). Global stabilization by output feedback: examples and counterexamples. *Systems and Control Letters* **23**, 119–125.

Medanić, J. (1967). Bounds on the performance index and the Riccati equation in differential games. *IEEE Transactions on Automatic Control* **12**, 613–614.

Megretski, A. (1996). $\mathcal{L}_2$ BIBO output feedback stabilization with saturated control. In: *Preprints of the 13th IFAC World Congress*. Vol. D. San Francisco, CA. pp. 435–440.

Megretski, A. and A. Rantzer (1997). System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control* **42**, 819–830.

Michel, A.N. and R.K. Miller (1977). *Qualitative Analysis of Large Scale Dynamical Systems*. Academic Press. New York.

Molander, P. and J.C. Willems (1980). Synthesis of state-feedback control laws with a specified gain and phase margin. *IEEE Transactions on Automatic Control* **25**, 928–931.

Monaco, S. and D. Normand-Cyrot (1986). Nonlinear systems in discrete-time. In: *Algebraic and Geometric Methods in Nonlinear Control Theory* (M. Fliess and M. Hazewinkel, Eds.). pp. 411–430.

Monaco, S. and D. Normand-Cyrot (1997). About nonlinear digital control. In: *Nonlinear Systems* (A.J. Fossard and D. Normand-Cyrot, Eds.). Vol. 3. pp. 127–153. Chapman & Hall. London.

Moore, F.K. and E.M. Greitzer (1986). A theory of post-stall transients in axial compression systems -Part I: Development of equations. *Journal of Turbomachinery* **108**, 68–76.

Moraal, P.E. and J.W. Grizzle (1995). Observer design for nonlinear systems with discrete-time measurements. *IEEE Transactions on Automatic Control* **40**, 395–404.

Moylan, P. (1974). Implications of passivity in a class of nonlinear systems. *IEEE Transactions on Automatic Control* **19**, 373–381.

Moylan, P.J. and B.D.O. Anderson (1973). Nonlinear regulator theory and an inverse optimal control problem. *IEEE Transactions on Automatic Control* **18**, 460–465.

Murray, R.M. (1995). Nonlinear control of mechanical systems:A Lagrangian perspective. In: *Preprints of the 3rd IFAC Nonlinear Control Systems Design Symposium*. Tahoe City, CA. pp. 378–389.

Narendra, K.S. and C.P. Neuman (1966). Stability of a class of differential equations with a single monotone nonlinearity. *SIAM Journal of Control and Optimization* **4**, 295–308.

Narendra, K.S. and J. Taylor (1973). *Frequency Domain Methods in Absolute Stability*. Academic Press. New York.

Narendra, K.S. and R.M. Goldwyn (1964). A geometrical criterion for the stability of certain nonlinear nonautonomous systems. *IEEE Transactions on Circuit Theory* **11**, 406–408.

Naumov, B.N. and Y.Z. Tsypkin (1965). A frequency criterion for absolute process stability in nonlinear automatic control systems. *Automation and Remote Control* **25**, 765–778. Translated from *Avtomatika i Telemekhanika*, 25:852-867, 1964.

Nešić, D. and I.M.Y. Mareels (1998). Dead beat controllability of polynomial systems: Symbolic computation approaches. *IEEE Transactions on Automatic Control* **43**, 162–176.

Nešić, D., A.R. Teel and P.V. Kokotović (1999). Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Systems and Control Letters* **38**, 259–270.

Nijmeijer, H. and A.J. van der Schaft (1990). *Nonlinear Dynamical Control Systems*. Springer-Verlag. New York.

Ortega, R. (1989). Passivity properties for stabilization of cascaded nonlinear systems. *Automatica* **27**, 423–424.

Ortega, R., A. Loría, H. Sira-Ramírez and P.J. Nicklasson (1998). *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Springer-Verlag. London.

Pan, Z. and T. Başar (1998). Adaptive controller design for tracking and disturbance attenuation in parametric strict-feedback nonlinear systems. *IEEE Transactions on Automatic Control* **43**, 1066–1084.

Pan, Z. and T. Başar (1999). Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion. *SIAM Journal of Control and Optimization* **37**, 957–995.

Park, P. and S.W. Kim (1998). A revisited Tsypkin criterion for discrete-time nonlinear Lur'e systems with monotonic sector-restrictions. *Automatica* **34**, 1417–1420.

Petersen, I.R. and B.R. Barmish (1987). Control effort considerations in the stabilization of uncertain dynamical systems. *Systems and Control Letters* **9**, 417–422.

Pomet, J.-B., R.M. Hirschorn and W.A. Cebuhar (1993). Dynamic output feedback regulation for a class of nonlinear systems. *Mathematics of Control, Signals, and Systems* **6**, 106–124.

Popov, V.M. (1960). Criterion of quality for non-linear controlled systems. In: *Preprints of the First IFAC World Congress*. Butterworths. Moscow. pp. 173–176.

Popov, V.M. (1962). Absolute stability of nonlinear systems of automatic control. *Automation and Remote Control* **22**, 857–875. Translated from *Avtomatika i Telemekhanika*, 22:961-979, 1961.

Popov, V.M. (1963). The solution of a new stability problem for controlled systems. *Automation and Remote Control* **24**, 1–23. Translated from *Avtomatika i Telemekhanika*, 24:7-26, 1963.

Praly, L. (1992). Lyapunov design of a dynamic output feedback for systems

linear in their unmeasured state components. In: *Preprints of the 2nd IFAC Nonlinear Control Systems Design Symposium.* Bordeaux, France. pp. 31–36.

Praly, L. and Y. Wang (1996). Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Mathematics of Control, Signals, and Systems* **9**, 1–33.

Praly, L. and Z.-P. Jiang (1993). Stabilization by output-feedback for systems with ISS inverse dynamics. *Systems and Control Letters* **21**, 19–33.

Praly, L. and Z.-P. Jiang (1998). Further results on robust semiglobal stabilization with dynamic input uncertainties. In: *Proceedings of the 37th IEEE Conference on Decision and Control.* Tampa, FL. pp. 891–897.

Praly, L., G. Bastin, J.-P. Pomet and Z.-P. Jiang (1991). Adaptive stabilization of nonlinear systems. In: *Foundations of Adaptive Control* (P.V. Kokotović, Ed.). pp. 347–435. Springer-Verlag. Berlin.

Protz, J. and J. Paduano (1997). Rotating stall and surge: Alternate modeling and control concepts. In: *Proceedings of the IEEE International Conference on Control Applications.* Hartford. pp. 866–873.

Raghavan, S. and J.K. Hedrick (1994). Observer design for a class of nonlinear systems. *International Journal of Control* **59**, 515–528.

Rajamani, R. (1998). Observers for Lipschitz nonlinear systems. *IEEE Transactions on Automatic Control* **43**, 397–401.

Rantzer, A. (1996). On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters* **28**, 7–10.

Saberi, A., P.V. Kokotović and H.J. Sussmann (1990). Global stabilization of partially linear composite systems. *SIAM Journal of Control and Optimization* **28**, 1491–1503.

Safonov, M.G. (1980). *Stability and Robustness of Multivariable Feedback Systems.* MIT Press. Cambridge, MA.

Safonov, M.G. and M. Athans (1977). Gain and phase margins for multiloop LQG regulators. *IEEE Transactions on Automatic Control* **22**, 173–179.

Sandberg, I.W. (1964*a*). A frequency domain condition for the stability of systems containing a single time-varying nonlinear element. *The Bell System Technical Journal* **43**, 1601–1638.

Sandberg, I.W. (1964*b*). On the $\mathcal{L}_2$-boundedness of solutions of nonlinear functional equations. *The Bell System Technical Journal* **43**, 1581–1599.

Sastry, S. (1999). *Nonlinear Systems: Analysis, Stability, and Control.* Springer-Verlag. New York.

Sastry, S., J. Hauser and P. Kokotović (1989). Zero dynamics of regularly perturbed systems may be singularly perturbed. *Systems and Control Letters* **13**, 299–314.

Sepulchre, R. (2000). Slow peaking and low-gain designs for global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* **45**, 453–461.

Sepulchre, R. and M. Arcak (1998). Global stabilization of nonlinear cascade systems: Limitations imposed by right half-plane zeros. In: *Preprints of the 4th IFAC Nonlinear Control Systems Design Symposium.* Enschede, Nether-

lands. pp. 624–630.

Sepulchre, R., M. Janković and P. Kokotović (1997). *Constructive Nonlinear Control*. Springer-Verlag. New York.

Seto, D., A.M. Annaswamy and J. Baillieul (1994). Adaptive control of nonlinear systems with a triangular structure. *IEEE Transactions on Automatic Control* **39**, 1411–1428.

Shewchun, J.M. and E. Feron (1997). High performance bounded control of systems subject to input and input rate constraints. *American Inst. of Aeronautics and Astronautics* **36**, 770–779.

Shouse, K.R. and D.G. Taylor (1998). Sensorless velocity control of permanent-magnet synchronous motors. *IEEE Transactions on Control Systems Technology* **6**, 313–324.

Šiljak, D.D. (1978). *Large-Scale Systems: Stability and Structure*. North Holland. New York.

Sira-Ramírez, H., M. Rios-Bolívar and A.S.I. Zinober (1997). Adaptive dynamical input-output linearization of DC-to-AC power converters: A backstepping approach. *International Journal of Robust and Nonlinear Control* **7**, 279–296.

Sontag, E.D. (1979). *Polynomial Response Maps*. Springer-Verlag. Berlin.

Sontag, E.D. (1983). A Lyapunov-like characterization of asymptotic controllability. *SIAM Journal of Control and Optimization* **21**, 462–471.

Sontag, E.D. (1989*a*). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control* **34**, 435–443.

Sontag, E.D. (1989*b*). A universal construction of Artstein's theorem on nonlinear stabilization. *Systems and Control Letters* **13**, 117–123.

Sontag, E.D. (1998*a*). Comments on integral variants of ISS. *Systems and Control Letters* **34**, 93–100.

Sontag, E.D. (1998*b*). *Mathematical Control Theory*. Vol. 6 of *Texts in Applied Mathematics*. second ed.. Springer-Verlag. New York.

Sontag, E.D. and H.J. Sussmann (1988). Further comments on the stabilizability on the angular velocity of a rigid body. *Systems and Control Letters* **12**, 437–442.

Sontag, E.D. and Y. Wang (1995). On characterizations of the input-to-state-stability property. *Systems and Control Letters* **24**, 351–359.

Sontag, E.D. and Y. Wang (1996). New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control* **41**, 1283–1294.

Strand, J.P., K. Ezal, T.I. Fossen and P.V. Kokotović (1998). Nonlinear control of ships: A locally optimal design. In: *Proceedings of the 4th IFAC Nonlinear Control Systems Design Symposium*. Enschede, Netherlands. pp. 732–737.

Sussmann, H.J. (1990). Limitations on the stabilizability of globally minimum phase systems. *IEEE Transactions on Automatic Control* **35**, 117–119.

Sussmann, H.J. and P.V. Kokotović (1991). The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* **36**, 424–439.

Sussmann, H.J., E.D. Sontag and Y. Yang (1994). A general result on the

stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control* **39**, 2411–2426.

Szegö, G. (1963). On the absolute stability of sampled-data control systems. *Proceedings of the National Academy of Sciences of the United States of America* **49**, 558–560.

Tao, G. and P.A. Ioannou (1988). Strictly positive real matrices and the Lefschetz-Kalman-Yakubovich lemma. *IEEE Transactions on Automatic Control* **33**, 1183–1185.

Teel, A.R. (1992). Using saturation to stabilize a class of single-input partially linear composite systems. In: *Preprints of the 2nd IFAC Nonlinear Control Systems Design Symposium*. Bordeaux, France. pp. 224–229.

Teel, A.R. (1996*a*). A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control* **41**(9), 1256–1271.

Teel, A.R. (1996*b*). On graphs, conic relations and input-output stability of nonlinear feedback systems. *IEEE Transactions on Automatic Control* **41**(5), 702–709.

Teel, A.R. (1998). A nonlinear control viewpoint on anti-windup and related problems. In: *Preprints of the 4th IFAC Nonlinear Control Systems Design Symposium*. Enschede, Netherlands. pp. 115–120.

Teel, A.R. and L. Praly (1994). Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems and Control Letters* **22**, 313–325.

Teel, A.R. and L. Praly (1995). Tools for semiglobal stabilization by partial state feedback and output feedback. *SIAM Journal of Control and Optimization* **33**, 1443–1488.

Teel, A.R. and L. Praly (2000). On assigning the derivative of a disturbance attenuation control Lyapunov function. *Mathematics of Control, Signals, and Systems* **13**, 95–124.

Teel, A.R. and N. Kapoor (1997). The $\mathcal{L}_2$ anti-windup problem: Its definition and solution. In: *Proceedings of the European Control Conference*.

Teel, A.R., D. Nešić and P.V. Kokotović (1998). A note on input-to-state stability of sampled-data nonlinear systems. In: *Proceedings of the 37th IEEE Conference on Decision and Control*. Tampa, FL. pp. 2473–2479.

Teel, A.R., T.T. Georgiou, L. Praly and E. Sontag (1996). Input-output stability. In: *The Control Handbook* (W.S. Levine, Ed.). pp. 895–908. CRC Press.

Tezcan, I.E. and T. Başar (1999). Disturbance attenuating adaptive controllers for parametric strict feedback nonlinear systems with output measurements. *ASME Journal on Dynamic Systems, Measurement and Control* **121**, 48–57.

Thau, F.E. (1973). Observing the state of non-linear dynamic systems. *International Journal of Control* **17**, 471–479.

Tsinias, J. (1989*a*). Observer design for nonlinear systems. *Systems and Control Letters* **13**, 135–142.

Tsinias, J. (1989*b*). Sufficient Lyapunov-like conditions for stabilization. *Math-*

ematics of Control, Signals, and Systems **2**, 343–357.

Tsinias, J. (1991). Existence of control Lyapunov functions and applications to state feedback stabilizability of nonlinear systems. *SIAM Journal of Control and Optimization* **29**, 457–473.

Tsitsiklis, J.N. and M. Athans (1984). Guaranteed robustness properties of multivariable nonlinear stochastic optimal regulators. *IEEE Transactions on Automatic Control* **29**, 690–696.

Tsypkin, Y.Z. (1962). The absolute stability of large-scale, nonlinear sampled-data systems. *Doklady Akademii Nauk SSSR* **145**, 52–55.

Tsypkin, Y.Z. (1963). Fundamentals of the theory of non-linear pulse control systems. In: *Preprints of the Second IFAC World Congress*. Balse, Switzerland. pp. 172–180.

Tsypkin, Y.Z. (1964). Absolute stability of equilibrium positions and of responses in nonlinear, sampled-data, automatic systems. *Automation and Remote Control* **24**, 1457–1470. Translated from *Avtomatika i Telemekhanika*, 24:1601-1615, 1963.

Tsypkin, Y.Z. (1965). Absolute stability of a class of nonlinear automatic sampled data systems. *Automation and Remote Control* **25**, 918–923. Translated from *Avtomatika i Telemekhanika*, 25:1030-1036, 1964.

Utkin, V.I. (1992). *Sliding Modes in Optimization and Control*. Springer-Verlag. New York.

van der Schaft, A.J. (1991). On a state space approach to nonlinear $\mathcal{H}_\infty$ control. *Systems and Control Letters* **16**, 1–8.

van der Schaft, A.J. (1992). $\mathcal{L}_2$ gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control. *IEEE Transactions on Automatic Control* **37**, 770–784.

van der Schaft, A.J. (1996). L_2-*Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag. New York.

van Nieuwstadt, M., P.E. Moraal, I.V. Kolmanovsky, A. Stefanopoulou, P. Wood and M. Criddle (1998). Decentralized and multivariable design for EGR-VGT control of a diesel engine. In: *IFAC Workshop on Advances in Automotive Control*. Mohican State Park, OH.

Vidyasagar, M. (1993). *Nonlinear Systems Analysis*. second ed.. Prentice Hall. Englewood Cliffs, New Jersey.

Wen, J.T. (1988). Time domain and frequency domain conditions for strict positive realness. *IEEE Transactions on Automatic Control* **33**, 988–992.

Willems, J.C. (1972). Dissipative dynamical systems Part I: General theory; Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis* **45**, 321–393.

Xiao, C. and D. Hill (1998). Concepts of strict positive realness and the absolute stability problem of continuous-time systems. *Automatica* **34**, 1071–1082.

Yakubovich, V.A. (1962). The solution of certain matrix inequalities in automatic control theory. *Doklady Akademii Nauk* **143**, 1304–1307.

Yakubovich, V.A. (1965). The matrix-inequality method in the theory of the

stability of nonlinear control systems-Parts I-III. *Automation and Remote Control.* Translated from *Avtomatika i Telemekhanika*, 25:1017-1029, 1964, 26:577-590, 26:753-763, 1965.

Yang, Y., E.D. Sontag and H.J. Sussmann (1997). Global stabilization of linear discrete-time systems with bounded feedback. *Systems and Control Letters* **30**, 273–281.

Yaz, E. (1993). Stabilizing compensator design for uncertain nonlinear systems. *Systems and Control Letters* **25**, 11–17.

Yoshizawa, T. (1966). *Stability Theory by Lyapunov's Second Method.* The Mathematical Society of Japan. Tokyo.

Zames, G. (1964). The input-output stability of nonlinear and time-varying feedback systems. In: *Proceedings of the National Electronics Conference.* pp. 725–730.

Zames, G. (1966). On the input-output stability of time-varying nonlinear feedback systems-Parts I and II. *IEEE Transactions on Automatic Control* **11**, 228–238 and 465–476.

Zames, G. (1981). Feedback and optimal sensitivity: Model reference transformation, multiplicative seminorms and approximate inverses. *IEEE Transactions on Automatic Control* **26**, 301–320.

Zames, G. and P.L. Falb (1968). Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal of Control and Optimization* **6**, 89–108.

Zhang, Y., P.A. Ioannou and C.-C. Chien (1996). Parameter convergence of a new class of adaptive controllers. *IEEE Transactions on Automatic Control* **41**, 1489–1493.

Zubov, V.I. (1957). *The Methods of A.M. Liapunov and their Application.* Leningrad University.

Zubov, V.I. (1966). *Theory of optimal control.* Sudostroenie, Leningrad.